SOLUTIONS EXERCISE SHEET 1

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1. Exercise 1

(a): We start with the claim that the Radon Nikodym derivative is a co-cycle:

Claim. The Radon-Nikodym derivative satisfies the following equation for μ-a.e. $x \in X$:

$$\frac{\mathrm{d}gh_*\mu}{\mathrm{d}\mu}(x) = \frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x)\frac{\mathrm{d}h_*\mu}{\mathrm{d}x}(g^{-1}\cdot x)$$

By uniqueness of the Radon Nikodym derivative and the Riesz representation theorem it suffices to show that:

$$\int_X \frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x)\frac{\mathrm{d}h_*\mu}{\mathrm{d}x}(g^{-1}\cdot x)f(x)\,\mathrm{d}\mu(x) = \int_X f(x)\,\mathrm{d}gh_*\mu(x) \quad \forall f \in \mathscr{C}_c(X)$$

We calculate:

$$\begin{aligned}\int_X \frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x)\frac{\mathrm{d}h_*\mu}{\mathrm{d}x}(g^{-1}\cdot x)f(x)\,\mathrm{d}\mu(x) &= \int_X \frac{\mathrm{d}h_*\mu}{\mathrm{d}\mu}(g^{-1}\cdot x)f(x)\,\mathrm{d}g_*\mu(x)\\ &= \int_X \frac{\mathrm{d}h_*\mu}{\mathrm{d}\mu}(x)f(g\cdot x)\,\mathrm{d}\mu(x)\\ &= \int_X f(g\cdot x)\,\mathrm{d}h_*\mu(x)\\ &= \int_X f(gh\cdot x)\,\mathrm{d}\mu(x)\end{aligned}$$

as desired. It follows that for almost every $x \in X$ holds:

$$\begin{aligned}\pi_{gh}(f)(x) =& \sqrt{\frac{\mathrm{d}gh_*}{\mathrm{d}\mu}(x)}f\big((gh)^{-1}\cdot x\big)\\ =& \sqrt{\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}(x)}\sqrt{\frac{\mathrm{d}h_*\mu}{\mathrm{d}\mu}(g^{-1}\cdot x)}f\big(h^{-1}\cdot(g^{-1}\cdot x)\big)\\ =& \pi_g\left(\sqrt{\frac{\mathrm{d}h_*\mu}{\mathrm{d}\mu}}f\circ h^{-1}\right)(x) = \pi_g(\pi_h f)(x)\end{aligned}$$

(b): For continuity we recall that by density and unitarity it suffices to check continuity of $g \mapsto \pi_g f$ for $f \in \mathscr{C}_c(X)$ at the identity. Denote $\gamma_g := \sqrt{\frac{\mathrm{d}g_*\mu}{\mathrm{d}\mu}}$ and calculate:

$$\begin{aligned}\|\pi_g f - f\|_2^2 =& \int_X \big|\gamma_g(x)f(g^{-1}\cdot x) - f(x)\big|^2\,\mathrm{d}\mu(x)\\ \leq& 2\int_X |\gamma_g(x)-1|^2\,\big|f(g^{-1}\cdot x)\big|^2\,\mathrm{d}\mu(x)\\ &+ 2\int_X \big|f(g^{-1}\cdot x) - f(x)\big|^2\,\mathrm{d}\mu(x)\end{aligned}$$

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As of equicontinuity at $1 \in G$, for every $\epsilon > 0$ there is a compact neighborhood V of $1 \in G$ such that:

$$|\gamma_g(x) - 1| < \sqrt{\epsilon} \quad \text{for } \mu\text{-a.e. } x \in X$$

and by continuity we know that:

$$\lim_{g \to 1} |\gamma_g(x) - 1|^2 \left|f(g^{-1} \cdot x)\right|^2 = 0 \quad \text{for } \mu\text{-a.e. } x \in X$$

By continuity of the group action the set $V \cdot \operatorname{supp} f$ is compact and hence $\phi := \|f\|_\infty^2 \mathbb{1}_{V \operatorname{supp} f} \in L^1_\mu(X)$ and for all $g \in V$ holds:

$$|\gamma_g(x) - 1|^2 \left|f(g^{-1} \cdot x)\right|^2 \leq \epsilon\phi(x) \quad \text{for } \mu\text{-a.e. } x \in X$$

Furthermore we know that:

$$\left|f(g^{-1} \cdot x) - f(x)\right|^2 \leq 2\left|f(g^{-1} \cdot x)\right|^2 + 2|f(x)|^2 \in L^1_\mu(G)$$

Hence we can apply Lebesgue's dominated convergence theorem to obtain:

$$\lim_{g \to 1} \|\pi_g f - f\|_2^2 = 0$$

2. Exercise 2

Let π be an irreducible, unitary representation of the abelian group G. Then for all $h \in G$ the following commutes:

$$\begin{CD} \mathcal{H}_\pi @>{\pi_g}>> \mathcal{H}_\pi \\ @V{\pi_h}VV @VV{\pi_h}V \\ \mathcal{H}_\pi @>>{\pi_g}> \mathcal{H}_\pi \end{CD} \qquad \forall g \in G$$

Thus Schur's lemma tells us that $\pi_h = \chi_h \operatorname{id}_{\mathcal{H}_\pi}$ for some character $\chi : G \to \mathbb{S}^1$. Hence every one-dimensional subspace of $\mathcal{H}_\pi$ is a π-invariant subspace and as of irreducibility $\mathcal{H}_\pi$ is one-dimensional.

3. Exercise 3

The regular representation reg of $\mathbb{R}$ is not norm-continuous. Let $t \in (0, \infty)$ and $f(x) := \sqrt{\frac{1}{2t}}\mathbb{1}_{[-t,t]}(x)$. Note that:

$$\left|\mathbb{1}_{[-t,t]}(x + 3t) - \mathbb{1}_{[-t,t]}(x)\right|^2 = \mathbb{1}_{[-t,t]}(x + 3t) + \mathbb{1}_{[-t,t]}(x)$$

and thus:

$$\|\operatorname{reg}_{3t} f - f\|_2^2 = \frac{1}{t}\int_{\mathbb{R}} \mathbb{1}_{[-t,t]}(x)\, \mathrm{d}x = 2$$

and hence $\|\operatorname{reg}_{3t} - \operatorname{reg}_0\| \geq \sqrt{2}$.

4. Exercise 4

Note that:

$$\int_G \int_G \left|\psi(x)\phi(x^{-1}g)\right| \mathrm{d}m_G(g)\, \mathrm{d}m_G(x) = \|\phi\|_1 \|\psi\|_1 < \infty$$

for all $\psi, \phi \in L^1(G)$ and thus by Fubini's theorem the convolution $\psi * \phi$ is well-defined m_G almost everywhere and in $L^1(G)$. The above calculation and Fubini's theorem implies continuity of the convolution:

$$\|\psi * \phi\|_1 \leq \|\phi\|_1 \|\psi\|_1$$

Given a unitary representation π of G, we know that $|\langle\pi_g v, w| \leq \|v\| \, \|w\|$ for all $v, w \in \mathcal{H}_\pi$ and hence $g \mapsto f(g)\langle\pi_g v, w\rangle \in L^1(G)$ whenever $f \in L^1(G)$. For $f \in L^1(G)$ we define $\pi(f) : \mathcal{H}_\pi \to \mathcal{H}_\pi$ by requiring:

$$\langle\pi(f)v, w\rangle = \int_G f(g)\langle\pi_g v, w\rangle \quad \forall v, w \in \mathcal{H}_\pi$$

which uniquely defines $\pi(f)$ by the Fréchet-Riesz representation theorem. It is clear that the map $f \mapsto \pi(f)$ is linear. Note that:

$$\|\pi(f)v\| = \sup_{\|w\| \leq 1} |\langle\pi(f)v, w\rangle| \leq \int_G |f| \, \|v\| \, \|w\| \, \mathrm{d}m_G \leq \|f\|_1 \, \|v\|$$

Hence $\|\pi(f)\| \leq \|f\|_1$ and hence $\pi : L^1(G) \to B(\mathcal{H}_\pi)$ is a bounded homomorphism of Banach spaces. We claim that in fact it is a homomorphism of Banach algebras. This is one long but simple calculation using Fubini:

$$\begin{aligned}
\langle\pi(\psi * \phi)v, w\rangle &= \int_G (\psi * \phi)(g)\langle\pi_g v, w\rangle \, \mathrm{d}m_G(g) \\
&= \int_G \psi(x) \left(\int_G \phi(x^{-1}g)\langle\pi_g v, w\rangle \, \mathrm{d}m_G(g) \right) \mathrm{d}m_G(x) \\
&= \int_G \psi(x) \left(\int_G \phi(g)\langle\pi_g v, \pi_{x^{-1}} w\rangle \, \mathrm{d}m_G(g) \right) \mathrm{d}m_G(x) \\
&= \int_G \psi(x)\langle\pi_x \pi(\phi)v, w\rangle \, \mathrm{d}m_G(x) \\
&= \langle\pi(\psi)\pi(\phi)v, w\rangle
\end{aligned}$$

which proves the claim.

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