SOLUTIONS EXERCISE SHEET 2

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Exercise 2

Let $\Phi : \mathcal{H}_\pi(v) \to L^2_{\mu_v}(\hat{G})$ be the isomorphism (of unitary representations) defined by $\Phi(v) := \mathbb{1}_{\hat{G}}$. Assume that $w \in \mathcal{H}_\pi(v)$ is a (normalized) eigenvector for G with eigenvalue $\chi_0 \in \hat{G}$. Then:

$$\chi_0(g) = \langle \pi_g w, w \rangle_\pi = \langle \pi_g \Phi(w), \Phi(w) \rangle_{L^2(\mu_v)} = \int_{\hat{G}} \langle g, \xi \rangle \left| \Phi(w)(\xi) \right|^2 \mathrm{d}\mu_v$$

The dual group is locally compact Hausdorff and its one-point compactifitcation is a weak* closed subset of the unit ball in $L^\infty(G)$, thus metrizable (under our standing assumptions). Because of the Hausdorff property, $\bigcap_{U \in \mathcal{U}_{\chi_0}} = \{\chi_0\}$, where $\mathcal{U}_{\chi_0}$ is a neighbourhood basis of χ_0 and by first countability we can assume that $\mathcal{U}$ is countable. Assuming $\mu_v(\{\chi_0\}) = 0$ hence yields a contradiction by outer regularity of μ_v.

Now assume that $\mu_v(\{\chi_0\}) > 0$, then $f := \mathbb{1}_{\{\chi_0\}} \in L^2_{\mu_v}(\hat{G})$ is a non-trivial eigenvector for the eigenvalue χ_0, hence so is $w := \Phi^{-1}(f) \in \mathcal{H}_\pi(v)$.

Exercise 5

Let π a finite dimensional, unitary representation of $G = \mathrm{SL}_2(\mathbb{R})$. As $\mathcal{H}_\pi$ is finite dimensional, strong continuity implies uniform continuity of the representation. By a property of Lie groups, we get that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\exp} & G \\ {\scriptstyle \mathrm{d}\pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathfrak{gl}(\mathcal{H}_\pi) & \xrightarrow[\exp]{} & \mathrm{GL}(\mathcal{H}_\pi) \end{array}$$

Proof using the Mautner phenomenon: Continuity implies smoothness and thus $\mathrm{d}\pi$ is a finite dimensional representation of $\mathfrak{sl}_2(\mathbb{R})$, so that there exists some non-trivial vector $v \in \mathcal{H}_\pi$ which is annihilated by $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Letting $u := \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$, we get:

$$\pi_u v = \pi_{\exp(X)} v = \exp(\mathrm{d}\pi_X) v$$

but $\exp(\mathrm{d}\pi_X) v = v$ and thus v is fixed under π_u. By the Mautner phenomenon $\mathbb{1} < \mathcal{H}_\pi$ and as of semicimplicity of finite dimensional representations of G, $\mathcal{H}_\pi$ must be a trivial representation.

Proof using Howe-Moore: Let π be a finite-dimensional, unitary representation of G. Let $E_\pi := \{v \in \mathcal{H}_\pi ; \pi_g v = v \forall g \in G\}$. Then $\mathcal{H}_\pi = E_\pi \oplus E_\pi^\perp$. We need to show that $E_\pi^\perp = \{0\}$. Assuming otherwise, fix $v \in E_\pi^\perp$ and assume that $\|v\| = 1$. Given $t \in \mathbb{R}$, denote $u_t := \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)$. There is some sequence $(n_k)_{k \in \mathbb{N}}$ such that $n_k \uparrow \infty$ and $\pi_{u_{n_k}} v$ converges to some $v^* \in E_\pi^\perp$. Let $v_k := \pi_{u_{n_k}}$. Using Howe-Moore:

$$1 = \|v^*\|^2 = \lim_{k \to \infty} \langle v_k, v^* \rangle = 0$$

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which is absurd.

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