SOLUTIONS EXERCISE SHEET 3

MANUEL W. LUETHI

Exercise 2

Amenability and (T) imply compactness: Assume that G is amenable and has property (T), then $\mathbb{1} < L^2(G)$ and in particular $L^2(G)$ has a non-trivial fixed vector.

Claim. Let $f \in L^2(G)$ invariant under G, then $|f|^2$ is a constant function.

Define the complex valued measure $m_f(E) := \int_E |f(x)|^2 \,\mathrm{d}x$, defined on all Borel subsets of G. By assumption on f, the measure m_f is a non-trivial Haar measure, and there exists a constant $c > 0$ such that $\mathrm{d}m_f(x) = c\,\mathrm{d}x$. By uniqueness of the Radon-Nikodym derivative (under our standing assumptions), it follows that $|f(x)|^2 = c$ almost everywhere. As $c > 0$, G necessarily has finite Haar measure. Hence G is compact.

Compactness implies amenability and (T). If G is compact, $\mathbb{1}_G \in L^2(G)$ and hence G is amenable. It remains to prove property (T). Assume that $\mathbb{1} \prec \pi$, then by compactness, for all $\epsilon > 0$ there exist vectors $v_1, \ldots, v_{n(\epsilon)} \in \mathcal{H}_\pi$ such that $1 = \sum_{i=1}^{n(\epsilon)} \|v_i\|^2 = 1$ and:

$$\sup_{g \in G} \left| 1 - \sum_{i=1}^{n(\epsilon)} \langle \pi_g v_i, v_i \rangle \right| < \epsilon$$

Assume that the Haar measure on G is normalized, i.e. $\mathrm{Vol}(G) = 1$. Let $v_1, \ldots, v_n$ as above for some constant ϵ, let $v := \sum_{i=1}^n v_i \in \mathcal{H}_\pi^{\oplus n}$ and define a vector $w \in \mathcal{H}_\pi^{\oplus n}$ by:

$$w = \int_G \pi_g^{\oplus n} v \,\mathrm{d}g$$

This is an invariant vector in $\mathcal{H}_\pi^{\oplus n}$ satisfying $|1 - \langle w, w\rangle| \leq \epsilon$ and thus for ϵ sufficiently small, w is a non-trivial fixed vector in $\mathcal{H}_\pi^{\oplus n}$. As the orthogonal projections $P_i : \mathcal{H}_\pi^{\oplus n} \to \mathcal{H}_\pi$, $(v_j)_{j=1}^n \mapsto v_i$ are all G-equivariant and because $w \neq 0$, there is some $1 \leq i \leq n$ such that $P_i(w) \neq 0$ is a non-trivial fixed vector in $\mathcal{H}_\pi$ and hence $\mathbb{1} < \mathcal{H}_\pi$.

Exercise 3

(a) Let $\Phi : \mathcal{H}_\pi \to \mathcal{H}_\pi^*$ be the anti-linear isomorphism obtained from Fréchet-Riesz. Let $v, w \in \mathcal{H}$. Then:

$$\overline{\pi}_g \Phi(v)(w) = \langle \pi_{g^{-1}} w, v\rangle_{\mathcal{H}_\pi} = \langle w, \pi_g v\rangle = \Phi(\pi_g v)(w)$$

As v, w were arbitrary, it follows that $\Phi \circ \pi_g = \overline{\pi}_g \circ \Phi$ for all $g \in G$. Hence:

$$\begin{aligned} \langle \overline{\pi}_g \Phi(v), \overline{\pi}_g \Phi(w)\rangle_{\mathcal{H}_\pi^*} &= \langle \Phi(\pi_g v), \Phi(\pi_g w)\rangle_{\mathcal{H}_\pi^*} = \langle \pi_g w, \pi_g v\rangle_{\mathcal{H}_\pi} \\ &= \langle w, v\rangle_{\mathcal{H}_\pi} = \langle \Phi(v), \Phi(w)\rangle_{\mathcal{H}_\pi^*} \end{aligned}$$

Date: April 10, 2016.

As Φ is onto, as clearly $\overline{\pi}_g\overline{\pi}_h = \overline{\pi}_{gh}$ and thus $\overline{\pi}_g$ clearly is a linear bijection, it follows that each $\overline{\pi}_g$ is unitary. It remains to show continuity of the orbit map. Let $v \in \mathcal{H}_\pi$, then:

$$\|\overline{\pi}_g\Phi(v) - \Phi(v)\|_{\mathcal{H}_\pi^*} = \|\Phi(\pi_g v - v)\|_{\mathcal{H}_\pi^*} = \|\pi_g v - v\|_{\mathcal{H}_\pi}$$

and thus $\overline{\pi}$ is a unitary representation of G.

(b) We prove that the regular representation λ of G is isomorphic to its contragredient $\overline{\lambda}$. To this end define a complex conjugation $\sigma : L^2(G) \to L^2(G)$ by $\sigma(f) = \overline{f}$ a.e. whenever $f \in L^2(G)$. Letting $\Phi : L^2(G) \to L^2(G)^*$ be the anti-linear isometry obtained via Fréchet-Riesz, the map $\Phi \circ \sigma$ is unitary:

$$\langle \Phi(\overline{f}), \Phi(\overline{h})\rangle_{L^2(G)^*} = \langle \overline{h}, \overline{f}\rangle_{L^2(G)} = \langle f, h\rangle_{L^2(G)}$$

It is obvious that σ is G-equivariant and hence as of what we have shown in part (a):

$$\begin{array}{ccc} L^2(G) & \xrightarrow{\Phi\circ\sigma} & L^2(G)^* \\ {\scriptstyle\lambda_g}\big\downarrow & & \big\downarrow{\scriptstyle\overline{\lambda}_g} \quad \forall g \in G \\ L^2(G) & \xrightarrow[\Phi\circ\sigma]{} & L^2(G)^* \end{array}$$

(c) The argument in part (b) did only rely on the existence of a G-equivariant, antilinear isometry of $L^2(G)$. Thus whenever one like this is available, the contragredient and the original representation are isomorphic. Note that in part (b) the G-equivariant, antilinear isometry was in fact an involution ant $L^2(G)$ has an invariant real form.

Exercise 4

The solution to part (a) is very long, not because the problem is so difficult, but because we present three different solutions, which show distinct properties of unitary representations at work.

(a) Let Dih_4 denote the dihedral group for the square, i.e. Dih_4 is the group generated by multiplication by $\mathbf{i}$ (denoted by r) and complex conjugation (denoted by s): $\mathrm{Dih}_4 = \langle r, s\rangle$. The natural action of this group on the two-dimensional real vector space $\mathbb{C}$ is real linear and preserves the inner product $(\cdot,\cdot)$ on $\mathbb{C} \cong \mathbb{R}^2$. In what follows, in order to reduce confusion, we denote by $\mathfrak{X}$ the real vector space $\mathbb{C}$ equipped with the real linear representation of r, s. The real linear representation extends to a unitary action on the complexification $\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C}$, equipped with the inner product given on simple tensors by:

$$\langle x \otimes \alpha, y \otimes \beta\rangle = (x, y)\alpha\overline{\beta} \quad x, y \in \mathfrak{X}, \alpha, \beta \in \mathbb{C}$$

As Dih_4 is discrete, the orbit maps are continuous and thus the representation is unitary. The orbit under Dih_4 of any non-trivial vector $x \in \mathfrak{X}$ generates $\mathfrak{X}$,as x and rx are linearly independent over $\mathbb{R}$. Thus the representation over $\mathfrak{X}$ is irreducible over $\mathbb{R}$.

Claim. The complexification of the representation is irreducible.[1]

[1] Note that the complexification of an irreducible representation over $\mathbb{R}$ need not be irreducible over $\mathbb{C}$. The representation of $\mathrm{SO}(2)$ over $\mathbb{R}^2$ is clearly irreducible, but its complexification on $\mathbb{C}^2$ (which is unitary) is not irreducible: As $\mathrm{SO}(2)$ is compact, we know that the representation decomposes as a direct sum of irreducible representations of $\mathrm{SO}(2)$ and because $\mathrm{SO}(2)$ is abelian, the irreducible subrepresentations are all one-dimensional.

Let $v \in \mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C}$ be any non-trivial vector and note that we can always write $v = x \otimes 1 + y \otimes \mathbf{i}$ with $x, y \in \mathfrak{X}$. It follows that $rv = rx \otimes 1 + ry \otimes \mathbf{i}$. Now assume that rv and v are linearly dependent, i.e. there is $\alpha \in \mathbb{C}$ such that $rv + \alpha v = 0$. Write $\alpha = a + \mathbf{i}b$, then:

$$0 = rv + \alpha v = (rx + ax - by) \otimes 1 + (ry + bx + ay) \otimes \mathbf{i}$$

Recall that $\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^2$ where the isomorphism sends $x \otimes 1 + y \otimes \mathbf{i}$ to (x, y), thus for any $w_1, w_2 \in \mathfrak{X}$ the vectors $w_1 \otimes 1$ and $w_2 \otimes \mathbf{i}$ are linearly independent, unless both are trivial. Thus $0 = rv + \alpha v$ implies that:

$$0 = rx + ax - by \qquad 0 = ry + bx + ay$$

If $b = 0$, then this implies $rx = -ax$ and $ry = -ay$, which means $x = y = 0$. So assume $b \neq 0$ and let $x' := b^{-1}x$, then $y = rx' + ax'$ and:

$$0 = r^2x' + bx + a^2x' + 2arx' = (b^2 + a^2 - 1)x' + 2arx'$$

Thus $a = 0$. But $a = 0$ yields $rx' = y$ and $ry = -bx$ and thus $x' = bx$. As $b \neq 0$, thus implies $x = 0$ and hence $y = 0$. Thus rv and v are linearly independent over $\mathbb{C}$ whenever v is non-trivial and as the complex dimension of $\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C}$ equals the real dimension of $\mathfrak{X}$, the representation is irreducible.

Claim. The sum $V := (\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C})$ is cyclic.

Let $1 \in \mathbb{C} \cong \mathfrak{X}$ be the first vector of the standard basis and consider the cyclic subrepresentation $W \subseteq V$ generated by $v := (1 \otimes 1, r1 \otimes 1)$. We claim that the orbit of v contains a basis of V so that $W = V$. We calculate explicitely:

$$\begin{aligned} v &= (1 \otimes 1, \mathbf{i} \otimes 1) \\ rv &= (\mathbf{i} \otimes 1, -1 \otimes 1) \\ sv &= (1 \otimes 1, -\mathbf{i} \otimes 1) \\ rsv &= (\mathbf{i} \otimes 1, 1 \otimes 1) \end{aligned}$$

As $\dim_{\mathbb{C}} V = 2 \dim_{\mathbb{C}}(\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C}) = 4$, it suffices to show that the above list is linearly independent. Note that for any two $x, y \in \mathfrak{X}$ and for any $\alpha = a + \mathbf{i}b \in \mathbb{C}$ holds:

$$\alpha(x \otimes 1, y \otimes 1) = a(x \otimes 1, y \otimes 1) + b(x \otimes \mathbf{i}, y \otimes \mathbf{i})$$

Hence for any finite collections $\{x_i\}_{i=1}^n$, $\{y_i\}_{i=1}^n$ in $\mathfrak{X}$, $\{\alpha_i = a_i + \mathbf{i}b_i\}_{i=1}^n$ in $\mathbb{C}$ holds:

$$\begin{aligned} & 0 = \sum_{i=1}^n \alpha_i (x_i \otimes 1, y_i \otimes 1) \\ \Leftrightarrow\ & 0 = \sum_{i=1}^n \big((a_i x_i) \otimes 1, (a_i y_i) \otimes 1\big) = \sum_{i=1}^n \big((b_i x_i) \otimes \mathbf{i}, (b_i y_i) \otimes \mathbf{i}\big) \\ \Leftrightarrow\ & 0 = \sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i y_i = \sum_{i=1}^n b_i x_i = \sum_{i=1}^n b_i y_i \end{aligned}$$

Thus it suffices to check that the vectors $(1, \mathbf{i})$, $(\mathbf{i}, -1)$, $(1, -\mathbf{i})$ and $(\mathbf{i}, 1)$ in $\mathfrak{X} \times \mathfrak{X}$ are linearly independent over $\mathbb{R}$, which reduces to an elementary calculation. Alternatively one checks that the representation $\mathfrak{X} \oplus \mathfrak{X}$ is cyclic (by the same argument) and that – unlike irreducibility – cyclicity is preserved under complexification which. This follows from the fact that any $\mathbb{R}$-basis $\{v_i\}_{i=1}^n$ of a real vector space V is mapped to a $\mathbb{C}$-basis of the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ under $v \mapsto v \otimes 1$.

Instead of proving that the sum $(\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C})$ is a cyclic representation – which is apparently a bit tedious – we could argue as follows: As Dih_4 is a finite group, the left-regular representation over $L^2(\mathrm{Dih}_4)$ is cyclic, as it is generated by the orbit of the Dirac function concentrated at the identity: $\delta_g = \lambda_g \delta_e$ for all $g \in \mathrm{Dih}_4$. Using Peter-Weyl, we know that:

$$L^2(\mathrm{Dih}_4) \cong \bigoplus_{[\pi] \in (\mathrm{Dih}_4)^{\hat{}}} (\dim \pi)\pi$$

and hence if Dih_4 has an irreducible representation of dimension at least 2, then we have shown the claim. But $\mathfrak{X} \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible with dimension 2 as argued above.

We provide another argument which relies on describing the unitary representations of Dih_4 which have dimension one. By the preceding description of $L^2(\mathrm{Dih}_4)$, we know that $\dim L^2(\mathrm{Dih}_4) = |\mathrm{Dih}_4| = 8$. Thus if we can show that the number of one-dimensional representations of Dih_4 is less than 8, the Peter-Weyl theorem tells us that there must be an irreducible representation of Dih_4 of dimension at least 2 and that it will have multiplicity greater than 1 in the cyclic representation $L^2(\mathrm{Dih}_4)$. Let $\chi : \mathrm{Dih}_4 \to \mathbb{S}^1$ be a character (i.e. a one-dimensional, unitary representation). Then $\chi(r)^4 = \chi(r^4) = 1$ and $\chi(s)^2 = \chi(s^2) = 1$. It follows that $\chi(s) \in \{\pm 1\}$ and $\chi(r) \in \{\pm 1, \pm \mathbf{i}\}$. Now note that for any $z \in \mathbb{C}$ holds $srs(z) = s(\mathbf{i}\overline{z}) = -\mathbf{i}z$ and thus $srs = r^{-1}$. Hence:

$$\overline{\chi(r)} = \chi(r^{-1}) = \chi(srs) = \chi(s)^2\chi(r) = \chi(r)$$

as $\chi(s) \in \{\pm 1\}$. Hence in fact $\chi(r) \in \{\pm 1\}$, implying that $|(\mathrm{Dih}_4)^{\hat{}}| \leq 4$. As $\dim L^2(\mathrm{Dih}_4) = 8$, we can deduce that there exists exactly one irreducible, unitary representation of Dih_4 of dimension more than 1 and indeed it must be the representation of dimension 2 desribed above.

(b) As discussed previously, if G is a finite group, then $L^2(G)$ is cyclic and by the Peter-Weyl theorem the multiplicity of $[\pi] \in \hat{G}$ inside $L^2(G)$ equals $\dim \pi$. This suggests the general upper bound $\dim \pi$ for the multiplicity of an irreducible representation in any cyclic representation. If $\dim \pi = \infty$, the boun clearly holds, so we only need to consider the case where $\dim \pi < \infty$.

Let ρ, π be cyclic representations of G and assume that π is irreducible and finite dimensional. As ρ is cyclic, so is the restriction of ρ to $\mathcal{H}_\rho^{[\pi]}$. For the sake of contradiction, we assume that the multiplicity of $[\pi]$ in ρ is strictly larger than d and after decomposing $\mathcal{H}_\rho^{[\pi]}$ into an orthogonal sum of representations in $[\pi]$ and then restricting to the first $d+1$ summands, we can without loss of generality assume that $\mathcal{H}_\rho^{[\pi]} \cong (\dim \pi + 1)\pi$. Under this assumption holds:

$$\begin{aligned} B_G(\rho, \pi) &= \Big\{ A \in B\big((\dim \pi + 1)\mathcal{H}_\pi, \mathcal{H}_\pi\big); A \circ \pi^{\oplus(d+1)} = \pi \circ A \Big\} \\ &\cong \bigoplus_{i=1}^{d+1} B_G(\pi, \pi) \cong \mathbb{C}^{d+1} \end{aligned}$$

On the other hand we know that $\mathcal{H}_\rho^{[\pi]}$ is cyclic, thus if we fix a generator v_0 of $\mathcal{H}_\rho^{[\pi]}$, then every equivariant map $A \in B_G(\rho, \pi)$ is completely determined by the value of $Av_0 \in \mathcal{H}_\pi$. In particular, the linear map $A \mapsto Av_0$ is injective, thus $B_G(\rho, \pi) \hookrightarrow \mathcal{H}_\pi$ and it follows that $\dim B_G(\rho, \pi) \leq d$, which contradicts the previous calculation.

E-mail address: `manuel.luethi@math.ethz.ch`