SOLUTIONS EXERCISE SHEET 4

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Exercise 1

Recall that:

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} ; |\alpha|^2 + |\beta|^2 = 1 \right\} \cong \mathbb{S}^3 \subseteq \mathbb{C}^2$$

with the homeomorphism $\psi : \mathbb{S}^3 \to \mathrm{SU}(2) =: G$ given by $(\alpha, \beta) \mapsto \left(\begin{smallmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{smallmatrix}\right)$. In particular, the volume measure on $\mathbb{S}^3$ maps to a finite measure which does not vanish on open subsets by definition of the volume form. Note that:

$$(\alpha, \beta) \underbrace{\begin{pmatrix} \delta & \eta \\ -\overline{\eta} & \overline{\delta} \end{pmatrix}}_{g} = (\alpha\delta - \beta\overline{\eta}, \alpha\eta + \beta\overline{\delta})$$

Hence:

$$\psi\big((\alpha, \beta)g\big) = \begin{pmatrix} \alpha\delta - \beta\overline{\eta} & \alpha\eta + \beta\overline{\delta} \\ -\overline{\alpha\eta} - \overline{\beta}\delta & \overline{\alpha\delta} - \overline{\beta}\eta \end{pmatrix} = \psi(\alpha, \beta)g$$

Note that G acts on $\mathbb{S}^3$ by isometries, as naturally $G \hookrightarrow \mathrm{SO}(4)$. Hence the natural action of G preserves the Lebesgue measure on $\mathbb{S}^3$ and thus the push forward of the Lebesgue measure on $\mathbb{S}^3$ under ψ is a right-invariant Haar measure on G and by compactness, it is also left-invariant. In particular, using elementary calculus, the Haar measure on G up to normalization is given by:

$$\int_G f(g)\,\mathrm{d}g = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} f_\psi\big(\cos\varphi + \mathbf{i}\sin\varphi\cos\varrho, e^{\mathbf{i}\vartheta}\sin\varphi\sin\varrho\big)\sin^2\varphi\sin\varrho\,\mathrm{d}\varphi\,\mathrm{d}\varrho\,\mathrm{d}\vartheta$$

for all $f \in \mathscr{C}(G)$, where $f_\psi := f \circ \psi$. Denote:

$$m(\varphi, \varrho, \vartheta) := \psi(\cos\varphi + \mathbf{i}\sin\varphi\cos\varrho, e^{\mathbf{i}\vartheta}\sin\varphi\sin\varrho)$$

One calculates:

$$\det\big(m(\varphi, \varrho, \vartheta) - X\mathbb{1}\big) = X^2 - 2\cos\varphi X + 1$$

Hence the eigenvalues of $m(\varphi, \varrho, \vartheta)$ are solely and completely determined by φ. By the spectral theorem for normal operators, every $g \in G$ is unitarily diagonalizable and in fact the unitary operator can be chosen in G. It follows that the conjugacy class of $m(\varphi, \varrho, \vartheta)$ is given by:

$$[m(\varphi, \varrho, \vartheta)] = \{m(\varphi, \varrho', \vartheta'); 0 \leq \varrho' \leq \pi, 0 \leq \vartheta' < 2\pi\}$$

and thus for any conjugation invariant $F \in \mathscr{C}(G)$ holds:

$$\int_G F(g)\,\mathrm{d}g = 4\pi\int_0^{\pi} F \circ \psi(e^{\mathbf{i}\varphi}, 0)\sin^2\varphi\,\mathrm{d}\varphi$$

In particular, the push-forward of the Haar measure on $G^\natural$ is given by:

$$\int_{G^\natural} f(x)\,\mathrm{d}x = 4\pi\int_0^{\pi} (f \circ q)\big(m(\varphi, 0, 0)\big)\sin^2\varphi\,\mathrm{d}\varphi$$

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where we denote by $q : G \to G^\natural$ the quotient map. The measure $\sin^2\varphi \,\mathrm{d}\varphi$ on $[0,\pi]$ is the so-called Sato-Tate measure. Note that the map from $[0,\pi]$ to $G^\natural$ sending φ to the conjugacy class of $m(\varphi,0,0)$ is a homeomorphism. Let $x,y \in G^\natural$ be distinct. Then x,y identify with two distinct pairs $\{\omega_x,\overline{\omega_x}\}$ and $\{\omega_y,\overline{\omega_y}\}$ in $\mathbb{S}^1$. Let $r : G^\natural \to \mathbb{R}$ denote the map sending a pair $\{\omega,\overline{\omega}\}$ to $r_{\{\omega,\overline{\omega}\}} := \Re\omega$. Choose $\epsilon > 0$ such that for $I_\epsilon\big(\{\omega,\overline{\omega}\}\big) := r^{-1}(r_{\{\omega,\overline{\omega}\}}-\epsilon, r_{\{\omega,\overline{\omega}\}}+\epsilon)$ holds $I_\epsilon(x)\cap I_\epsilon(y) = \emptyset$. If we can show that $r^{-1}(a,b)$ is open in $G^\natural$ for any two real numbers $a<b$, then $G^\natural$ is Hausdorff and the map $\varphi \mapsto m(\varphi,0,0)$ is a homeomorphism. Let $g = \left(\begin{smallmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{smallmatrix}\right) \in G$. Then $\det(g - X\mathbb{1}) = X^2 - 2\Re\alpha X + 1$ as argued before. Hence:

$$(r\circ q)^{-1}(a,b) = \left\{\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix}; |\alpha|^2+|\beta|^2 = 1, a < \Re\alpha < b\right\}$$

which is clearly open in G, proving that $I_\epsilon(x)$ and $I_\epsilon(y)$ are disjoint neighbourhoods of x and y respectively.

Let π be an irreducible representation of G, then there is $n\in\mathbb{N}$ such that:

$$\mathcal{H}_\pi \cong \big\{f : \mathbb{C}^2 \to \mathbb{C}^2; f \text{ is homogeneous of degree } n\big\} =: V_n$$

and the representation is given by precomposition with the inverse. The space $\mathcal{H}_\pi$ has dimension $n+1$ and a basis given by the maps:

$$(u,v) \mapsto f_k(u,v) := u^k v^{n-k} \quad 0 \le k \le n$$

and in order to determine $\operatorname{tr}\pi_g$ for $g\in G$, it suffices to determine $\operatorname{tr}\pi_{m_\varphi}$ for $\varphi\in[0,\pi]$ and $m_\varphi := m(\varphi,0,0)$. Using the basis from above, this calculation is easy:

$$\pi_{m_\varphi} f_k(u,v) = f_k(e^{-\mathbf{i}\varphi}u, e^{\mathbf{i}\varphi}v) = e^{-\mathbf{i}k\varphi}e^{\mathbf{i}(n-k)\varphi} f_k(u,v)$$

It follows that:

$$\operatorname{tr}\pi_{m_\varphi} = \sum_{k=0}^n e^{\mathbf{i}(n-2k)\varphi} = \sum_{k=0}^n \cos\big((n-2k)\varphi\big)$$

Exercise 2

(a) Let $E \subseteq \mathfrak{g}$ be a basis such that v has weak partial derivatives for all $x\in E$. We have to show that the partial derivatives $\pi(x)v$ exist for all $x\in E$. Let v_x be the weak partial derivative of v for $x\in E$.

Claim. For all $t\in\mathbb{R}$ holds:

$$\pi_{\exp tx}v - v = \int_0^t \pi_{\exp sx} v_x \,\mathrm{d}s$$

Using Fréchet-Riesz and applying the definition of the integral, we have to show that on a dense subset $D\subseteq\mathcal{H}_\pi$ holds:

$$\langle \pi_{\exp tx}v - v, w\rangle = \int_0^t \langle \pi_{\exp sx} v_x, w\rangle \,\mathrm{d}s$$

Let $D := \mathscr{C}_\pi^\infty$ be the dense set of smooth vectors. Let $\varphi_w(s) = \langle \pi_{\exp sx}v, w\rangle$. Using unitarity of π, one easily checks that for any $\mathscr{C}_\pi^1$ vector $v\in\mathcal{H}$ and for any $x\in\mathfrak{g}$ holds:

$$\lim_{h\to 0}\frac{1}{h}(\pi_{\exp hx}v - v) = \lim_{h\to 0}\frac{1}{-h}(\pi_{\exp(-hx)}v - v)$$

This implies:

$$\lim_{h\to 0}\frac{\varphi_w(s+h)-\varphi_w(s)}{h} = -\langle \pi_{\exp sx}v, \pi(x)w\rangle = \langle \pi_{\exp sx}v_x, w\rangle$$

and thus $\varphi_w \in \mathscr{C}^1(\mathbb{R})$. In particular we get $\varphi_w(t) - \varphi_w(0) = \int_0^t \varphi_w'(s)\,\mathrm{d}s$ by the fundamental theorem of calculus and the claim follows. An application of the Cauchy-Schwarz inequality yields:

$$\lim_{t\to 0} \frac{\pi_{\exp tx} v - v}{t} = v_x$$

(b) We have shown in class that any tuple $(v, w) \in \mathcal{H}_\pi \times \mathcal{H}_\pi^n$ in the closure of the graph of T satisfies:

$$\langle w_j, u\rangle = -\langle v, \pi(x_j)u\rangle \quad \forall u \in \mathscr{C}_\pi^1$$

where $x_1, \ldots, x_j$ form a basis of $\mathfrak{g}$. In particular v has weak derivatives for a basis of $\mathfrak{g}$ and thus $v \in \mathscr{C}_\pi^1$. Hence $(v, w) \in \mathrm{Graph}(T)$ and thus $\overline{\mathrm{Graph}(T)} \subseteq \mathrm{Graph}(T)$, implying that T is a closed operator.

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