SOLUTIONS EXERCISE SHEET 5

MANUEL W. LUETHI

Exercise 1

Let $G = \mathrm{SL}_3(\mathbb{R})$, $f \in \mathscr{C}_c^\infty(G)$ positive such that $\int_G f = 1$. Assume that π is a representation without fixed vectors and $\mathbb{1} \prec \pi$. We consider the self-adjoint operator $T = \frac{1}{2}\pi(f)^*(\pi_{a_t} + \pi_{a_{-t}})\pi(f)$, where:

$$a_t := \begin{pmatrix} e^{-\frac{t}{2}} & & \\ & e^{\frac{t}{2}} & \\ & & 1 \end{pmatrix}$$

Recall that $\|T\| = \sup\{|\langle Tv, v\rangle|; v \in \mathcal{H}_\pi : \|v\| \leq 1\}$. On the other hand we know that $\pi(f)^* = \pi(f^*)$ with $f^*(g) := \overline{f(g^{-1})}$, where we use that G is unimodular. One calculates for arbitrary $g \in G$:

$$\begin{aligned}
\langle \pi(f)^*\pi_g\pi(f)v, w\rangle =& \langle \pi(f)v, \pi_g^{-1}\pi(f)w\rangle = \int_G f(x)\langle \pi_x v, \pi_{g^{-1}}\pi(f)w\rangle \,\mathrm{d}x \\
=& \int_G f(g^{-1}x)\langle \pi_x v, \pi(f)w\rangle \,\mathrm{d}x = \langle \pi(\lambda_g f)v, \pi(f)w\rangle \\
=& \langle \pi(f)^*\pi(\lambda_g f)v, w\rangle = \langle \pi(f^* * \lambda_g f)v, w\rangle
\end{aligned}$$

In particular $T = \pi(h)$ for $h = \frac{1}{2}(f^* * \lambda_{a_t} f + f^* * \lambda_{a_{-t}} f)$. Let $K := \operatorname{supp} h$, then K is compact. Hence there are $v_j \in \mathcal{H}_\pi$, $1 \leq j \leq n$, such that $\sum_{j=1}^n \|v_j\|^2 = 1$ and:

$$\left|1 - \sum_{j=1}^n \langle \pi_g v_j, v_j\rangle\right| < \frac{1}{10}$$

Note that:

$$\int_G f^* * \lambda_g f(x) = \left(\int_G f(x)\,\mathrm{d}x\right)^2 = 1 \Rightarrow \int_G h(g)\,\mathrm{d}g = 1$$

It follows, that for $\rho(g) := 1 - \sum_{j=1}^n \langle \pi_g v_j, v_j\rangle$ holds:

$$\begin{aligned}
1 =& \int_K h(g)\,\mathrm{d}g \\
\leq& \left|\int_K \sum_{j=1}^n h(g)\langle \pi_g v_j, v_j\rangle \,\mathrm{d}g\right| + \int_K h(g)\,|\rho(g)|\,\mathrm{d}g \leq \sum_{j=1}^n |\langle T v_j, v_j\rangle| + \frac{1}{10} \\
\leq& \frac{1}{2}\sum_{j=1}^n \left(|\langle \pi_{a_t}\pi(f)v_j, \pi(f)v_j\rangle| + |\langle \pi_{a_{-t}}\pi(f)v_j, \pi(f)v_j\rangle|\right) + \frac{1}{10} \\
\ll& \sum_{j=1}^n e^{-\frac{3|t|}{16}} S(\pi(f)v_j)^2 + \frac{1}{10}
\end{aligned}$$

Date: May 27, 2016.

where in the last step we used the assumption that $1 \not< \pi$. Let $k := \dim \mathfrak{sl}_3(\mathbb{R})$ and $X_1, \ldots, X_k$ a basis of $\mathfrak{sl}_3(\mathbb{R})$ for which:

$$S_1(w)^2 \asymp \|w\|^2 + \sum_{l=1}^{k} \|\pi(X_l)w\|^2 \quad \forall w \in \mathscr{C}_\pi^1$$

As we caluclated in class, this yields:

$$\begin{aligned} S_1\big(\pi(f)v_j\big)^2 &= \|\pi(f)v\|^2 + \sum_{l=1}^{k} \|\pi(X_l)\pi(f)v_j\|^2 = \|\pi(f)v_j\|^2 + \sum_{l=1}^{k} \|\pi(\partial_{X_l} f)v_j\|^2 \\ &\leq \|\pi(f)v_j\|^2 + \sum_{l=1}^{k} \|\pi(\partial_{X_l} f)v_j\|^2 \leq \underbrace{\left(\|f\|_1^2 + \sum_{l=1}^{k} \|\partial_{X_l} f\|_1^2 \right)}_{C_f} \|v_j\|^2 \end{aligned}$$

so that $\sum_{j=1}^n \|v_j\|^2 = 1$ implies:

$$1 \ll C_f e^{-\frac{3|t|}{16}} + \frac{1}{10}$$

For large t, this is absurd and thus $\mathbb{1} < \pi$.

Exercise 2

Claim. $\mathrm{SL}_2(\mathbb{F}_p)$ is generated by unipotents. More explicitely for every matrix $g \in \mathrm{SL}_2(\mathbb{F}_p)$ there exist $s_1, s_2, t_1, t_2 \in \mathbb{F}_p$ such that:

$$g = u_{s_1}^+ u_{t_1} u_{s_2}^+ u_{t_2}$$

where $u_s^+ := \left(\begin{smallmatrix} 1 & \\ s & 1 \end{smallmatrix}\right)$ and $u_t := \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)$.

In what follows, we denote $U^+ := \{u_s^+; s \in \mathbb{F}_p\}$ and $U = \{u_t; t \in \mathbb{F}_p\}$. First we note that for $s, t \in \mathbb{F}_p$ holds:

$$u_s^+ u_t = \begin{pmatrix} 1 & t \\ s & 1 + st \end{pmatrix}$$

thus it suffices to show that for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{F}_p)$ there is a product h of unipotents (i.e. of matrices in $U^+ \cup U$) such that $hg = \left(\begin{smallmatrix} 1 & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. Assume that $a \neq 1$. If $c \neq 0$, then $h' = 1$ and $h = u_{c^{-1}(1-a)}$ will do. Otherwise we know that $a \neq 0$ and thus this argument can be applied to $u_1^+ g$.

Solution 1. Let $G = \mathrm{SL}_2(\mathbb{F}_p)$ and π a non-trivial, unitary, irreducible representation of G. As argued in the second solution, either the upper or the lower unipotent subgroup acts non-trivially, so that we will for simplicity assume that $A = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$, which generates the upper unipotents, acts non-trivially. If instead the lower unipotents act non-trivially, the following argument can be adapted easily. Note that $A^p = 1$, thus the eigenvalues of π_A are contained in the group μ_p of p-th roots of unity. Furthermore we note that for $a \in \mathbb{F}_p^\times$ and $m \in \mathbb{N}$ satisfying $m \equiv a^2 \mod p$ holds:

$$A^m = \begin{pmatrix} 1 & a^2 \\ & 1 \end{pmatrix} = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} A \begin{pmatrix} a^{-1} & \\ & a \end{pmatrix}$$

and thus the eigenvalues of π_A are permuted under the maps $x \mapsto x^m$ for $m \in \mathbb{N}$ with $m \equiv a^2 \mod p$ for some a. Let λ be an eigenvalue of π_A. Then there is some non-trivial eigenvector $v \in \mathcal{H}_\pi$ with $\pi_A v = \lambda v$ and thus $A^m v = \lambda^m v$. Thus λ^m is an eigenvalue of A. If A has a non-trivial root of unity as eigenvalue, because the order of π_A is p and because the order of any non-trivial p-th root of unity equals p, the claim follows.

Claim. Let $g \in \mathrm{GL}_d(\mathbb{C})$ and assume that g has finite order. Then A is diagonalizable.

Over $\mathbb{C}$, the matrix g is similar to (a choice of) its Jordan normal form Λ_g. Assume that $g^k = \mathbb{1}$, then $\Lambda_g^k = \mathbb{1}$ and thus each Jordan block $J_{i,g}$ in Λ_g satisfies $J_{i,g}^k = \mathbb{1}$. A simple calculation shows that any Jordan block J satisfying $J^k = \mathbb{1}$ for some $k \in \mathbb{N}$ necessarily is contained in $\mathrm{GL}_1(\mathbb{C})$.
As π_A is non-trivial, it follows that π_A indeed has a non-trivial eigenvalue and hence we are done.

Solution 2. This second solution is in essence the same as the previous one but formulated using more spectral theory. Again we will assume that the group $U = \{\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right); x \in \mathbb{F}_p\} \leq \mathrm{SL}_2(\mathbb{F}_p)$ acts non-trivially. As argued before, the image π_U is jointly diagonalizable and thus the spectral decomposition

$$\mathcal{H}_\pi = \bigoplus_{i=1}^n V_i$$

with $V_i \subseteq \mathcal{H}_\pi$ one-dimensional and U invariant admits the choice of an index $1 \leq i \leq n$, and some non-trivial character $\chi \in \hat{\mathbb{F}}_p$, such that for $v \in V_i \setminus \{0\}$ and $u \in U$ holds $uv = \chi(u)v$, where we made the identification $U \cong \mathbb{F}_p$. Note that the character χ is injective because p is prime. Let $x \in \mathbb{F}_p^\times$ and denote $a(x) := \left(\begin{smallmatrix} x & \\ & x^{-1} \end{smallmatrix}\right)$. It follows that:

$$\pi_{u_t}(\pi_{a(x)}v) = \pi_{a(x)}(\pi_{a(x)^{-1}u_t a(x)}v) = \pi_{a(x)}(\pi_{u_{x^{-2}t}}v) = \chi(x^{-2}t)\pi_{a(x)}v$$

Thus $\pi_{a(x)}$ is again an eigenvector for U for the character $\chi \times x^{-2}$. As χ is injective and x^{-2} a unit, it follows that $\chi \circ x^{-2} = \chi \circ y^{-2}$ iff $x^2 = y^2$. In particular, the set $\left\{\pi_{a(x)}v; x \in \mathbb{F}_p^\times\right\}$ consists of eigenvectors for at least $\frac{p-1}{2} = \left|\left\{x^2; x \in \mathbb{F}_p^\times\right\}\right|$ many distinct eigenvalues of U (for $p \leq 3$ the statement in the exercise is trivially true) and hence we are done.

References

E-mail address: `manuel.luethi@math.ethz.ch`