TOPICS IN MATHEMATICAL AND COMPUTATIONAL FLUID DYNAMICS

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1 Fundamental Equations of Fluid Dynamics

The fundamental equations of fluid dynamics are derived under the so-called *continuum assumption*: since fluids, i.e., gases and liquids, usually consist of a very large number of particles, we model such fluids as a continuum rather than consider the motion of individual fluid particles.

The quantities of interest in such a continuum description are then macroscopic variables that can be measured point-wise. Key quantities of interest are, for example, the density, velocity, pressure, temperature and energy. A fluid model describes the coupled evolution of these variables in space and time.

1.1 Eulerian Description

Let $\Omega \subseteq \mathbb{R}^d$ ($d = 1, 2, 3$) be an open, bounded set with a smooth boundary denoted by $\partial\Omega$ (see Figure 1.1).

Let $x \in \Omega$ denote any point in the domain and let $t \in \mathbb{R}_+$ denote a point in time. Then, fluid models typically describe the evolution of

1. Density: $\rho(x, t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}$,
2. Velocity: $u(x, t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$,
3. Pressure: $p(x, t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}$,
4. Temperature: $\theta(x, t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}$,
5. Total Energy: $E(x, t) : \Omega \times \mathbb{R}_+ \to \mathbb{R}$,

and other, similar macroscopic quantities of interest.

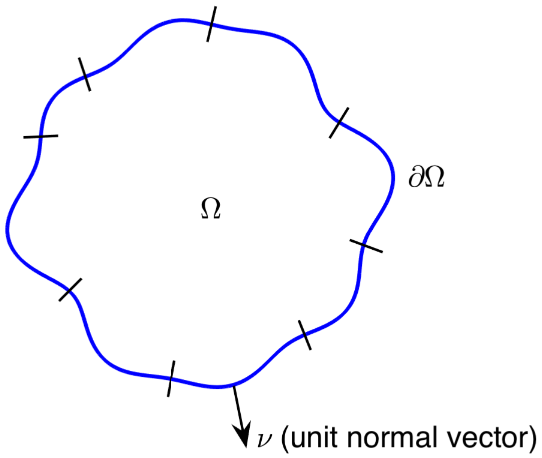


Figure 1.1: Example of the domain set Ω with smooth boundary $\partial\Omega$.

The space-time coordinates (x, t) are called *Eulerian coordinates* and an Eulerian fluid description consists of measuring quantities of interest at fixed Eulerian coordinates. Thus, for example, we can measure the wind velocity at a fixed point using an anemometer.

1.2 Lagrangian Description

The Eulerian description of fluid dynamics relies on a fixed frame of reference. In practice, however, we are primarily concerned with fluid flows. In such a situation, each fluid particle moves during the fluid evolution with its motion being modulated by the velocity u. An alternative approach, therefore, is to describe the motion of a continuum such as a fluid using a *flow mapping*

$$\mathcal{X} \colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d.$$

In particular, let $a \in \Omega$ be any point. Then the motion of this point (see Figure 1.2) can be expressed by the mapping

$$x = \mathcal{X}(a, t).$$

The point $a \in \Omega$ is then termed a *Lagrangian* or *material* coordinate, and we use the convention:

$$\mathcal{X}(a, 0) = a.$$

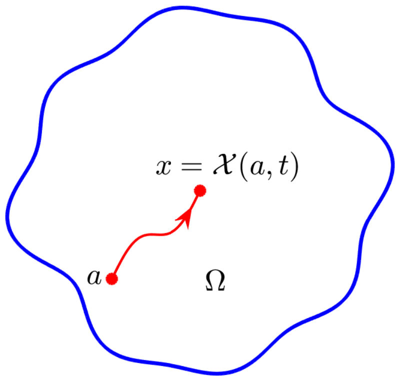


Figure 1.2: Example of the motion of the point $a \in \Omega$ under the mapping $\mathcal{X}$.

We assume that the function $\mathcal{X}$ is smooth in both arguments, and is also a diffeomorphism, i.e., $\mathcal{X}^{-1}$ exists and is differentiable. Furthermore, we assume that the mapping $\mathrm{D}_a^x \in \mathbb{R}^{d\times d}$ exists and is non-singular where

$$(\mathrm{D}_a^x)_{i,j} = \frac{\partial x_i}{\partial a_j}, \quad \forall\ i,j = 1,\dots,d.$$

Intuitively, this implies that the flow map $\mathcal{X}$ does not cause fluid volume of a non-zero measure to evolve to one of measure zero. Under these assumptions, the velocity U of a material particle a at time t is given by

$$U(a,t) = \frac{\partial \mathcal{X}}{\partial t}(a,t).$$

The Eulerian and Lagrangian descriptions of the velocity field are then related by the equation

$$\begin{aligned} x &= \mathcal{X}(a,t), \\ u\big(\mathcal{X}(a,t),t\big) &= U(a,t). \end{aligned} \tag{1.1}$$

Conversely, given a smooth Eulerian velocity field $u(x,t)$, we can obtain the Lagrangian mapping $\mathcal{X}$ by solving the following initial value problem:

$$\begin{aligned} \frac{d\mathcal{X}}{dt}(a,t) &= u\big(\mathcal{X}(a,t),t\big), \\ \mathcal{X}(a,0) &= a. \end{aligned} \tag{1.2}$$

Note that Equation (1.2) is a non-linear ODE and the solution to this IVP might result in extremely complicated ('chaotic') particle paths.

Next, given a function $g\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ in Eulerian coordinates, we can derive an analogous function $G\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ in Lagrangian coordinates by setting

$$G(a,t) := g\big(\mathcal{X}(a,t),t\big). \tag{1.3}$$

We denote the rate of change of g (for a fixed Eulerian coordinate $x \in \Omega$) with respect to time as $g_t(x,t)$ or $\partial_t g(x,t)$. Similarly, we denote the rate of change of the function G (for a fixed Lagrangian coordinate $a \in \Omega$) with respect to time as $\partial_t G := \frac{\mathrm{D}g}{\mathrm{D}t}$. We remark this quantity is also known as the *material derivative.*

It then follows that

$$\begin{aligned} \frac{\mathrm{D}g}{\mathrm{D}t}\big(\mathcal{X}(a,t),t\big) &= \partial_t g(x,t) + \nabla g(x,t) \cdot \frac{d\mathcal{X}}{dt} \\ &= \partial_t g + u \cdot \nabla g. \end{aligned} \tag{1.4}$$

Thus, Equation (1.4) provides a link between the Eulerian and Lagrangian descriptions of fluid flow. Intuitively, the material derivative

$$\frac{\mathrm{D}}{\mathrm{D}t} := \frac{\partial}{\partial t} + u \cdot \nabla,$$

describes the rate of change of some physical quantity along particle paths.

1.3 Reynolds Transport Theorem

Let $\mathcal{P} \subset \Omega$ be a bounded set (see Figure 1.3). For every $t \in \mathbb{R}_+$, we define the set

$$\mathcal{P}_t = \big\{\mathcal{X}(a,t)\colon a \in \mathcal{P}\big\}$$

We can now state the following generalisation of the Leibniz Rule, which is also known as differentiation under the integral sign.

Theorem 1.1 (Reynolds Transport Theorem) *Let $\mathcal{X}\colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ be a smooth function such that for all $t \in \mathbb{R}_+$ it holds that $\mathcal{X}(\cdot,t)$ is a diffeomorphism of $\mathbb{R}^d$, let the sets $\mathcal{P}, \mathcal{P}_t$ be defined as above and let $g\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be a smooth function in both arguments. Then it holds that*

$$\frac{d}{dt}\int_{\mathcal{P}_t} g(x,t)dx = \int_{\mathcal{P}_t} \partial_t g(x,t)dx + \int_{\partial\mathcal{P}_t} (u\cdot\nu) g(x,t) d\sigma(x). \tag{1.5}$$

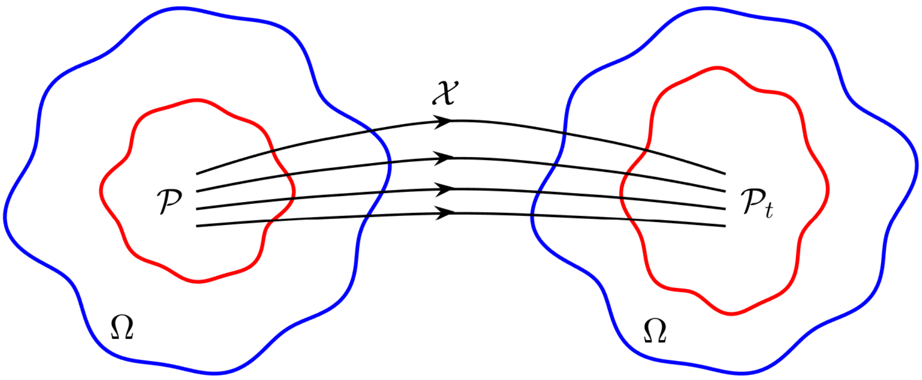


Figure 1.3: Example of the motion of the point $a \in \Omega$ under the mapping $\mathcal{X}$.

Here, $\partial\mathcal{P}_t$ is the boundary of the set $\mathcal{P}_t$, $\nu(x)$ is the unit outward normal vector at the point $x \in \partial\mathcal{P}_t$ and u is the Eulerian velocity field defined by Equation (1.1)*.*

We remark that applying the Divergence theorem to (1.5) results in the following equation:

$$\frac{d}{dt}\int_{\mathcal{P}_t} g(x,t)dx = \int_{\mathcal{P}_t} \partial_t g(x,t)dx + \int_{\mathcal{P}_t} \operatorname{div}\big(u(x,t)g(x,t)\big)dx. \tag{1.6}$$

1.4 Conservation Laws

The fundamental equations of fluid dynamics are often derived in terms of the following conservation laws:

1.4.1 Conservation of Mass

Let $\rho\colon \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ denote the fluid density. Then the total mass contained in any material volume $\mathcal{P}_t$ is given by

$$\int_{\mathcal{P}_t} \rho(x,t)dx.$$

Conservation of mass implies that

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho(x,t)dx = 0.$$

Reynolds Transport Theorem (1.6) then implies that

$$\int_{\mathcal{P}_t} \big(\partial_t \rho + \operatorname{div}(\rho u)\big)dx = 0. \tag{1.7}$$

Since Equation (1.7) holds for any material volume $\mathcal{P}_t$, the following point-wise equation must hold:

$$\rho_t + \operatorname{div}(\rho u) = 0. \tag{1.8}$$

1.4.2 Conservation of Momentum

The total momentum of any material volume $\mathcal{P}_t$ is given by

$$\int_{\mathcal{P}_t} \rho u \, dx.$$

Newton's Second Law of Motion then implies that

$$\frac{d}{dt}\int_{\mathcal{P}_t} \rho u \, dx = \int_{\mathcal{P}_t} Q \, dx, \tag{1.9}$$

where Q denotes the sum of all forces acting on the fluid.

Applying Reynolds Transport Theorem (1.6) component-wise to Equation (1.9) then yields

$$\int_{\mathcal{P}_t} \big(\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - Q\big)dx = 0. \tag{1.10}$$

Once again, since the above equation holds for any material volume $\mathcal{P}_t$, we obtain the following point-wise equation:

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) = Q. \tag{1.11}$$

Here, $(\cdot \otimes \cdot)$ denotes the tensor product. In particular, in the case of three spatial dimensions, i.e., $u \in \mathbb{R}^3$ given by $u = (u_1, u_2, u_3)$, it holds that

$$u \otimes u = \begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2^2 & u_2u_3 \\ u_1u_3 & u_2u_3 & u_3^2 \end{bmatrix}.$$

It remains to specify the total force Q. Note that we may write the total force as

$$Q = F + S,$$

where F represents so-called body forces that act on the entire volume of the fluid, and S represents so-called surface forces that are internal forces acting on the surface of the material volume. In the absence of body forces such as gravity, buoyancy or the Coriolis force, we may assume that $F \equiv 0$ and therefore, we need only model the surface forces S.

Remark 1.2 *We may use the mass conservation equation* (1.8) *in Equation* (1.11) *to obtain*

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = S.$$

The above equation can then be rewritten in terms of the material derivative as

$$\rho \frac{Du}{Dt} \equiv S. \tag{1.12}$$

Equation (1.12) *is simply the more explicit form of Newton's Second Law of Motion, which, informally, states that mass times acceleration equals the force. This formulation of the conservation of momentum equation* (1.11) *is occasionally useful.*

The surface forces S can now be computed using the so-called *Cauchy momentum tensor*:

$$S = \operatorname{div} \sigma,$$

where $\sigma \in \mathbb{R}^{d \times d}$ is the stress (force per unit surface area) tensor and $\operatorname{div} \sigma$ is the component-wise divergence of the stress tensor. In particular, in the case of three spatial dimensions, the stress tensor σ is denoted by

$$\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy} & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix},$$

where σ_{ii} denotes the Normal stress in the i-direction and τ_{ij} denotes the shear stress across the i and j directions.

Next, we define the pressure p as the mean normal stress given by

$$p = -\frac{1}{3}(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}).$$

The stress tensor can then be rewritten as

$$\sigma = -\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} \sigma_{xx}+p & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_{yy}+p & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz}+p \end{bmatrix}$$
$$= -p\mathbb{I} + \mathrm{T}.$$

Here, $\mathbb{I}$ is the 3×3 Identity matrix and T is the so-called *deviatoric stress tensor*. Note that T has trace zero.

Using the above notations and definitions, Equation (1.11) can be written in the form

$$\begin{aligned} (\rho u)_t + \operatorname{div}(\rho u \otimes u) &= -\nabla p + \operatorname{div} \mathrm{T} \\ \implies (\rho u)_t + \operatorname{div}(\rho u \otimes u + p\mathbb{I}) &= \operatorname{div} \mathrm{T}. \end{aligned} \tag{1.13}$$

It now remains to specify the deviatoric stress tensor T. For simplicity, we restrict ourselves to the case of a *Newtonian fluid*. This allows us to make the following assumptions:

1. The stress is linearly proportional to the strain, i.e, $\mathrm{T} \propto \nabla u$.
2. The fluid under consideration is isotropic, i.e., the mechanical properties of the fluid are invariant under rotations.
3. The fluid is in hydrostatic equilibrium at rest, i.e., $u \equiv 0 \implies \mathrm{T} \equiv 0$.

Under these assumptions, the most general form of the deviatoric stress tensor is given by

$$\mathrm{T} = \mu\big(\nabla u + (\nabla u)^{\intercal}\big) + \lambda\mathbb{I}(\operatorname{div} u). \tag{1.14}$$

Here, $\nabla u \in \mathbb{R}^{d\times d}$ is the tensor derivative of the velocity vector field u, μ is the kinematic viscosity of the fluid and λ is the bulk viscosity of the fluid. It is customary to set $\lambda = -\frac{2}{3}\mu$.

1.4.3 Conservation of Energy

We define the energy density $E\colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ of the fluid as

$$E = E(x,t) := \underbrace{\rho(x,t)|u(x,t)|^2}_{\text{Kinetic Energy}} + \underbrace{\rho(x,t)\mathrm{e}(x,t)}_{\text{Internal Energy}},$$

where $e\colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ is the internal energy per unit volume of the fluid.

It is possible to use the conservation of momentum equation (1.13) to derive an expression for the evolution of the kinetic energy. Similarly, it is also possible to use either the enthalpy balance or the entropy balance, together with energy loss due to heat conduction to derive an expression for the evolution of the internal energy. For the sake of brevity, we skip these calculations.

The conservation of energy equation is then given by

$$E_t + \operatorname{div}\big((E+p)u\big) = \operatorname{div}(\mathrm{T}u) + \operatorname{div}(\kappa \nabla \theta), \tag{1.15}$$

where T is the deviatoric stress tensor (1.14), $\kappa \in \mathbb{R}$ is the coefficient of heat conduction and $\theta\colon \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ is the temperature.

We remark that the internal energy $e = e(\rho, p, \theta)$ is usually described by a so-called *equation of state*, which is derived from thermodynamics. In particular, in the case of an ideal gas, it holds that

$$\begin{aligned} e &= \frac{p}{(\gamma - 1)\rho}, \\ p &= \rho R \theta, \end{aligned}$$

where R is the universal gas constant and γ is another constant.

Equations (1.8), (1.13) and (1.15) together constitute the fundamental *compressible Navier-Stokes equations of fluid dynamics*. Furthermore, in the special case of an ideal gas we obtain the following complete system of equations:

$$\begin{aligned} \rho_t + \operatorname{div}(\rho u) &= 0, && \text{(Conservation of Mass)} \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u + p\mathbb{I}) &= \operatorname{div} \mathrm{T}, && \text{(Conservation of Momentum)} \\ E_t + \operatorname{div}\big((E+p)u\big) &= \operatorname{div}(\mathrm{T}u) + \operatorname{div}(\kappa \nabla \theta), && \text{(Conservation of Energy)} \end{aligned}$$

with

$$\begin{aligned} \mathrm{T} &= \mu\big(\nabla u + (\nabla u)^{\mathsf{T}}\big) + \lambda \mathbb{I}(\operatorname{div} u), && \text{(Newtonian Fluid)} \\ E &= \frac{1}{2}\rho |u|^2 + \frac{p}{\gamma - 1}, && \text{(Ideal Gas Equation of State)} \\ \theta &= \frac{p}{\rho R}. && \text{(Temperature Law)} \end{aligned} \tag{\textbf{CNS}}$$

1.5 Limiting Regime I: Inviscid Limit

The compressible Navier-Stokes Equations (**CNS**) relate quantities expressed in physical units. In order to understand and identify the different scales in the problem, it is customary to non-dimensionalise the equations. Doing so introduces a dimensionless constant known as the *Reynolds number* Re given by

$$\mathrm{Re} = \frac{\mathrm{UL}}{\mu},$$

where L is a typical length scale, U is a velocity scale and μ is the kinematic viscosity. The Reynolds number Re is the ratio of inertial forces to viscous forces and essentially quantifies the relative importance of these forces for a given flow.

In many flows of interest, $\mathrm{Re} \gg 1$ or equivalently $\mu \ll 1$. Furthermore the heat conduction coefficient κ is small for several fluids such as, for example, air. Hence, we can assume that $\mu = \kappa = 0$. Then, using the fact that $\lambda = \frac{2}{3}\mu$, we can reduce the compressible Navier-Stokes equations (**CNS**) to the so-called *compressible Euler equations*:

$$\begin{aligned}
\rho_t + \mathrm{div}(\rho u) &= 0, && \text{(Conservation of Mass)}\\
(\rho u)_t + \mathrm{div}(\rho u \otimes u + p\mathbb{I}) &= 0, && \text{(Conservation of Momentum)}\\
E_t + \mathrm{div}\left((E+p)u\right) &= 0 && \text{(Conservation of Energy)}
\end{aligned}$$

with

$$E = \frac{1}{2}\rho|u|^2 + \frac{p}{\gamma - 1}. \qquad \text{(Ideal Gas Equation of State)} \qquad (\mathbf{CE})$$

The compressible Euler equations (**CE**) are a prototypical example of a system of conservation laws:

$$\partial_t u + \mathrm{div}\left(f(u)\right) = 0, \tag{1.16}$$

and represent the fundamental equations of *inviscid fluid dynamics*.

1.6 Limiting Regime II: Incompressible Limit

May fluids of interest such as, for example, water in the ocean, are incompressible. For such fluids, the fluid density of an infinitesimal fluid volume remains constant along the flow. Mathematically, this amounts to the condition that the material derivative of the density is zero:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \rho_t + (u \cdot \nabla)\rho \equiv 0. \tag{1.17}$$

Applying (1.17) to the mass conservation equation in (**CE**) results in

$$\rho \operatorname{div} u = 0.$$

And since the density ρ is non-zero, the incompressibility condition implies the following divergence constraint:

$$\operatorname{div} u \equiv 0. \tag{1.18}$$

Next, we apply the divergence constraint (1.18) to the momentum conservation equation in (**CE**) and perform some manipulations using the chain rule to obtain

$$u_t + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = 0. \tag{1.19}$$

Furthermore, it is straightforward to show that the energy conservation equation in (**CE**) is redundant in the incompressible limit. Thus, setting $\rho \equiv 1$, we obtain the so-called *incompressible Euler equations*:

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= 0, \\ \operatorname{div} u &= 0. \end{aligned} \tag{\textbf{ICE}}$$

Finally, we can also reintroduce viscosity in the incompressible Euler equations (**ICE**) and make use of the divergence constraint (1.18) to simplify the stress tensor considerably to obtain the so-called *incompressible Navier-Stokes* equations:

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= \mu \Delta u, \\ \operatorname{div} u &= 0. \end{aligned} \tag{\textbf{INS}}$$

Note that the above derivation of the incompressible Navier-Stokes equations is heuristic. A more formal derivation is based on non-dimensionalising the compressible Navier-Stokes equations (**CNS**) by scaling it with the Mach number

$$\mathrm{Ma} = \frac{u}{a},$$

and deriving the zero Mach number limit. Here, a is the speed of sound in the fluid and is given by $a^2 = \frac{p}{\gamma\rho}$.

2 The Incompressible Navier-Stokes Equations

We recall from Chapter 1 that the incompressible Navier-Stokes equations for a Newtonian fluid (**INS**) are given by

$$u_t + (u \cdot \nabla)u + \nabla p = \mu \Delta u, \tag{2.1a}$$
$$\operatorname{div} u = 0. \tag{2.1b}$$

Here, $d \in \{2, 3\}$, $\mu \in \mathbb{R}$ is the kinematic viscosity, $u \in \mathbb{R}^d$ and $\nabla p \in \mathbb{R}^d$ are vectors given by

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix}, \qquad \nabla p = \begin{bmatrix} \partial_{x_1} p \\ \partial_{x_2} p \\ \vdots \\ \partial_{x_d} p \end{bmatrix},$$

$\operatorname{div} u$ and Δu are the divergence and Laplacian respectively of the velocity field u and are given by

$$\operatorname{div} u = \sum_{j=1}^{d} \frac{\partial u}{\partial x_j}, \qquad \Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2},$$

the vector operator $(u \cdot \nabla)u \in \mathbb{R}^d$ is given by

$$(u \cdot \nabla)u = \begin{bmatrix} \sum_{i=1}^{d} u_i \partial_{x_i} u_1 \\ \sum_{i=1}^{d} u_i \partial_{x_i} u_2 \\ \vdots \\ \sum_{i=1}^{d} u_i \partial_{x_i} u_d \end{bmatrix},$$

and we have used the notation $u_t := \frac{\partial u}{\partial t}$, $\partial_{x_i} p := \frac{\partial p}{\partial x_i}$ and $\partial_{x_i} u_j := \frac{\partial u_j}{\partial x_i}$.

We recall that the mass and momentum conservation equations of the compressible Navier-Stokes equations (**CNS**) are given by

$$\begin{aligned}\rho_t + \operatorname{div}(\rho u) &= 0,\\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p &= \operatorname{div}\big(\mu(\nabla u + (\nabla u)^T) + \lambda \operatorname{div} u \mathbb{I}\big).\end{aligned}$$

The incompressible limit can then formally be obtained by scaling these equations with the Mach number, deriving the zero Mach number limit, and setting $\rho \equiv 1$.

Properties of the Incompressible Navier-Stokes Equations (2.1)

- Equation (2.1) is a non-linear partial differential equation with both convective and diffusive terms.
- Equation (2.1) is a 2^{nd} order PDE due to the presence of a diffusive term.
- Solutions to Equation (2.1) satisfy a divergence constraint, that is, $\operatorname{div} u = 0$.
- Existence and uniqueness of solutions to Equation (2.1) in a completely general setting is an open question.

In order to analyse solutions to the incompressible Navier-Stokes equations, we must supplement Equation (2.1) with appropriate initial and boundary conditions. We therefore consider the following so-called *initial boundary value problem*:

Initial Boundary Value Problem for Equation (2.1)

Let $T \in (0, \infty]$, $d \in \{2, 3\}$ and let $\Omega \subset \mathbb{R}^d$ be a bounded, open set with smooth boundary $\partial\Omega$, let $u\colon [0,T) \times \Omega \to \mathbb{R}^d$ be the unknown velocity field and let $u_0\colon \Omega \to \mathbb{R}^d$ be a function. Then, consider the following initial boundary value problem (IBVP) for the incompressible Navier-Stokes equations (2.1)

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= \mu \Delta u, && \text{for } (t,x) \in (0,T) \times \Omega,\\ \operatorname{div} u &= 0, && \text{for } (t,x) \in (0,T) \times \Omega,\\ u(0,x) &= u_0(x), && \text{for } x \in \Omega,\\ u(t,x) &= 0, && \text{for } (t,x) \in (0,T) \times \partial\Omega. \end{aligned} \tag{2.2}$$

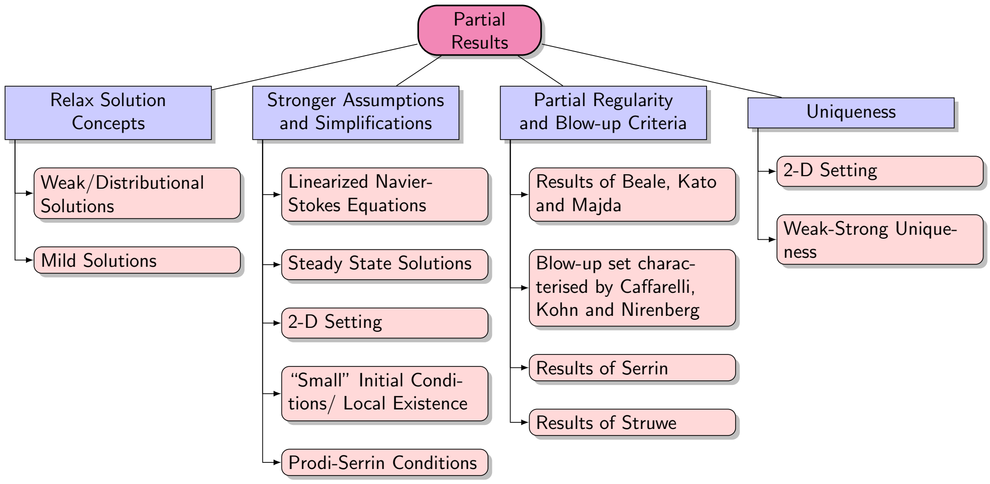


Figure 2.1: An incomplete overview of various research directions and results for the incompressible Navier-Stokes equations.

Question: Consider the IBVP (2.2) and let the initial function u_0 be smooth. Then does there exist a unique, global in time (i.e., $T = \infty$), classical, i.e., at least twice-continuously differentiable solution to Equation (2.2)?

Of course, the answer to this question is highly non-trivial. Indeed, the Clay Mathematics Institute in May 2000, set this problem (for dimension $d = 3$) as one of the seven Millennium Prize problems in mathematics. The Institute offers a prize of US $\$1,000,000$ to the first person providing a solution to any one of four specific statements of the above problem. For instance [Fef06],

Prove or give a counter-example of the following statement:

Take $\mu > 0$ and $d = 3$. Let $u_0(x)$ be any smooth, divergence-free vector field with the property that for any multi-index α and any constant K it holds that

$$|\partial_x^\alpha u_0(x)| \leq C_{\alpha,K}(1 + |x|)^{-K}.$$

Then there exist infinitely smooth functions $p := p(t,x) \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ and $u := u(t,x) \colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the initial value problem (2.2) and with the property that for all time $t > 0$ it holds that $\|u(\cdot,t)\|_{L^2}$.

Despite the difficulty of the problem and the lack of a complete solution, there are, nevertheless, some partial answers to the question of global existence and uniqueness of solutions to the incompressible Navier-Stokes equations. Figure 2.1 displays a broad (incomplete) overview of various research directions and results. An concise review of the current state-of-the-art and a list of references for further review can, for instance, be found in [Fef06].

2.1 Formal Calculations

Throughout this section, unless stated otherwise, we assume that all involved functions exist and are sufficiently smooth so as to allow the necessary manipulations.

2.1.1 Energy

Consider Equation (2.1a). We take the inner product with the function u on both sides of the equation and integrate over the domain Ω to obtain

$$\underbrace{\int_\Omega u \cdot u_t \, dx}_{\text{(i)}} + \underbrace{\int_\Omega u \cdot \big((u \cdot \nabla)u\big) dx}_{\text{(ii)}} + \underbrace{\int_\Omega u \cdot \nabla p \, dx}_{\text{(iii)}} = \underbrace{\mu \int_\Omega u \cdot \Delta u \, dx}_{\text{(iv)}} .$$

We now consider each term (i)-(iv) separately:

(i)
$$\int_\Omega u \cdot u_t \, dx = \frac{1}{2} \frac{d}{dt} \int_\Omega |u|^2 dx.$$

(ii)
$$\begin{aligned}\int_\Omega u \cdot \big((u \cdot \nabla)u\big) dx &= \int_\Omega \sum_{i=1}^d u_i \big(\sum_{j=1}^d u_j \partial_{x_j} u_i \big) dx = \sum_{i,j=1}^d \int_\Omega u_j \underbrace{\partial_{x_j} u_i u_i}_{=\frac{1}{2}\partial_{x_j}\left(|u_i|^2\right)} dx \\ &= \frac{1}{2} \sum_{i,j=1}^d \int_\Omega u_j \partial_{x_j} \big(|u_i|^2\big) dx \\ &\underbrace{=}_{\substack{\text{Integration} \\ \text{by parts}}} -\frac{1}{2} \sum_{i,j=1}^d \int_\Omega \partial_{x_j} \big(u_j |u_i|^2\big) dx = -\frac{1}{2} \int_\Omega (\operatorname{div} u) |u|^2 dx \\ &\underbrace{=}_{\text{Eq. (2.1b)}} 0.\end{aligned}$$

(iii)

$$\int_\Omega u \cdot \nabla p \, dx \underbrace{=}_{\substack{\text{Integration}\\ \text{by parts}}} - \int_\Omega (\operatorname{div} u) p \, dx$$
$$\underbrace{=}_{\text{Eq. (2.1b)}} 0.$$

(iv)

$$\mu \int_\Omega u \cdot \Delta u \, dx \underbrace{=}_{\substack{\text{Integration}\\ \text{by parts}}} -\mu \int_\Omega |\nabla u|^2 dx.$$

It therefore holds that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \mu \int_\Omega |\nabla u|^2 dx = 0, \tag{2.3a}$$

and integrating Equation (2.3a) in time thus yields

$$\frac{1}{2}\int_\Omega |u(t,x)|^2 dx + \mu \int_0^t \int_\Omega |\nabla u|^2(s,x)\, dx ds = \frac{1}{2}\int_\Omega |u_0(t,x)|^2 dx. \tag{2.3b}$$

Equation (2.3b) now implies that the total energy of the system is non-increasing in time:

$$E(t) := \int_\Omega |u(t,x)|^2 dx \leq \int_\Omega |u_0(x)|^2 dx = E(0).$$

2.1.2 Pressure

Consider Equation (2.1a). We apply the divergence operator on both sides of the equation to obtain

$$\partial_t \underbrace{(\operatorname{div} u)}_{=0.} + \operatorname{div}\big((u \cdot \nabla)u\big) + \Delta p = \mu \Delta \underbrace{(\operatorname{div} u)}_{=0.}.$$

Equation (2.1b) implies that the velocity field vector is divergence-free, and therefore

$$-\Delta p = \operatorname{div}\big((u \cdot \nabla)u\big) = \operatorname{div}\big(\operatorname{div}(u \otimes u)\big). \tag{2.4}$$

Equation (2.4) can be further simplified by observing that

$$\begin{aligned}\operatorname{div}\big((u \cdot \nabla)u\big) &= \sum_{i,j=1}^d \partial_{x_j}(u_i \partial_{x_i} u_j) = \sum_{i,j=1}^d \big(\partial_{x_j}\partial_{x_i}(u_i u_j) - \partial_{x_j}(u_j \partial_{x_i} u_i)\big)\\ &= \sum_{i,j=1}^d \partial_{x_j}\big(\partial_{x_i}(u_i u_j)\big) - \underbrace{\sum_{i,j=1}^d \partial_{x_j}\big(u_j \partial_{x_i} u_i\big)}_{=0 \text{ since } \operatorname{div} u = 0.},\end{aligned}$$

and therefore, the pressure field satisfies the following equation:

$$\Delta p = -\sum_{i,j=1}^{d} \partial_{x_j}\big(\partial_{x_i}(u_i u_j)\big). \tag{2.5}$$

Equation (2.5) is a Poisson equation for the pressure field. Since Equation (2.5) is elliptic, this implies that the pressure field for the incompressible Navier-Stokes equations is instantaneously determined by the velocity vector field and is no longer an independent variable. Of course, the solution of Equation (2.5) will necessitate the imposition of suitable boundary conditions.

2.1.3 Vorticity

We begin by defining the vorticity of a velocity vector field.

Definition 2.1 (Vorticity) *Let $T \in (0, \infty]$, $d \in \{2, 3\}$, and let $\Omega \subset \mathbb{R}^d$ be a bounded, open set and let $u\colon [0, T) \times \Omega \to \mathbb{R}^d$ be a velocity vector field. Then the vorticity ω of the velocity u is defined as $\omega = \operatorname{curl} u$. In particular*

$$d = 2 \implies \mathbb{R} \ni \omega = \operatorname{curl} u = \partial_{x_1} u_2 - \partial_{x_2} u_1,$$

$$d = 3 \implies \mathbb{R}^3 \ni \omega = \operatorname{curl} u = \begin{bmatrix} \partial_{x_2} u_3 - \partial_{x_3} u_2 \\ \partial_{x_3} u_1 - \partial_{x_1} u_3 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{bmatrix}.$$

Remark 2.2 *Let, $T \in (0, \infty]$, let $\Omega \subset \mathbb{R}$ be a bounded, open set and let $u\colon [0, T) \times \Omega \to \mathbb{R}$ be a velocity scalar field. We define the rotated gradient operator $\underline{\operatorname{curl}} u$ as*

$$\underline{\operatorname{curl}} u = \nabla^\perp u = \begin{bmatrix} -\partial_{x_2} u \\ \partial_{x_1} u \end{bmatrix}.$$

Using the definition of the vorticity, it is possible to rewrite the incompressible Navier-Stokes equation (2.1) as an equation involving the vorticity ω of the velocity field:

Proposition 2.3 *Consider the incompressible Navier-Stokes equation* (2.1)*, let $d \in \{2, 3\}$, let $u\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be a strong solution to Equation* (2.1) *and let $\omega = \operatorname{curl} u$ be the vorticity. Then it holds that*

$$d = 2 \implies \partial_t \omega + (u \cdot \nabla)\omega = \mu \Delta \omega, \tag{2.6a}$$

$$d = 3 \implies \partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla) u = \mu \Delta \omega. \tag{2.6b}$$

Proof The proof is left as an exercise. □

We end this section by stating some remarks regarding Equations (2.6).

Remark 2.4 *The two-dimensional vorticity equation* (2.6a) *is simply the heat equation with an additional convective term, and further reduces to a transport equation if the kinematic viscosity* $\mu = 0$*. Therefore, classical solutions of Equation* (2.6a) *satisfy both a maximum principle as well as* L^p *bounds. In particular, for any* $p \in [1, \infty)$*, multiplying both sides of Equation* (2.6a) *with* $\omega \cdot |\omega|^{p-2}$ *and integrating over the spatial domain yields the* L^p*-estimate*

$$\|\omega(t, \cdot)\|_{L^p} \leq \|\omega(0, \cdot)\|_{L^p}.$$

These bounds can be used to prove the existence of smooth solutions to the incompressible Navier Stokes equation (2.2) *in two spatial dimensions. Unfortunately, the same strategy does not work in three spatial dimensions. Indeed, the vortex-stretching term* $(\omega \cdot \nabla)u$ *considerably complicates the analysis.*

2.2 Leray-Hopf Solutions

The incompressible Navier-Stokes equations (2.1) have been extensively studied and there exists a vast literature on results pertaining to existence, uniqueness and regularity of both weak and strong solutions to these equations. Seminal contributions were first made by Jean Leray who, in [Ler34], constructed a global (in time) weak solution and a local strong solution of the IVP (2.2) in the special case $\Omega = \mathbb{R}^3$. Furthermore, Heinz Hopf proved in [Hop51], the existence of a global weak solution of the IBVP (2.2) with bounded domain. Since then, several mathematicians have studied the uniqueness and regularity of such weak solutions but there still remain many interesting open problems. In particular, the uniqueness and regularity of so-called Leray-Hopf Solutions in three spatial dimensions is currently unknown.

We begin by defining weak (distributional) solutions to the IBVP (2.2). Informally,

- We say that the function $u \in C^2\big([0, T) \times \Omega\big)$ is a *classical solution* to IBVP (2.2) if it satisfies Equation (2.2) point-wise.
- We say that the weakly differentiable function $u\colon [0, T) \times \Omega \to \mathbb{R}^d$ is a *distributional solution* to IBVP (2.2) if it satisfies Equation (2.2) in the

sense of distributions, i.e., Equation (2.2) holds after multiplication with a suitable test function and integrating over space and time.

We now attempt to make these ideas more precise.

Let $\phi \in C_c^\infty\big([0,\infty) \times \Omega; \mathbb{R}^d\big)$ be a function with the property that $\operatorname{div} \phi = 0$. Consider Equation (2.1a); we take the inner product with the function ϕ on both sides of the equation and integrate over the domain $[0,T) \times \Omega$ to obtain

$$\int_0^\infty \int_\Omega \phi \cdot \partial_t u + \phi \cdot \big((u \cdot \nabla)u\big) + \phi \cdot \nabla p \, dxdt = \mu \int_0^t \int_\Omega \phi \cdot \Delta u \, dxdt.$$

Then, using integration by parts and the fact that the test function ϕ vanishes at infinity, we obtain

$$\int_0^\infty \int_\Omega u \cdot \partial_t \phi + \big((u \cdot \nabla)\phi\big) \cdot u \, dxdt = \mu \int_0^t \int_\Omega \nabla \phi : \nabla u \, dxdt - \int_\Omega u_0 \, \phi(0,x) dx. \tag{2.7}$$

Definition 2.5 *Let $u \colon [0,T) \times \Omega \to \mathbb{R}^d$ be a function with the property that for all divergence-free test functions $\phi \in C_c^\infty\big([0,\infty) \times \Omega; \mathbb{R}^d\big)$ it holds that*

$$\int_0^\infty \int_\Omega u \cdot \partial_t \phi + \big((u \cdot \nabla)\phi\big) \cdot u \, dxdt = \mu \int_0^t \int_\Omega \nabla \phi : \nabla u \, dxdt - \int_\Omega u_0 \phi(0,x) dx.$$

Then we say that u solves the incompressible Navier-Stokes equation (2.1a) *in the sense of distributions.*

Similarly, let $\widetilde{\phi} \in C_c^\infty\big(\Omega\big)$ be a scalar test function. Consider Equation (2.1b); we multiply both sides of the equation with $\widetilde{\phi}$ and and integrate over the spatial domain Ω to obtain

$$\int_\Omega \widetilde{\phi} \operatorname{div} u \, dx = 0.$$

Once again, using integration by parts we obtain

$$\int_\Omega u \cdot \nabla \widetilde{\phi} \, dx = 0. \tag{2.8}$$

Definition 2.6 *Let* $u\colon [0,T)\times\Omega \to \mathbb{R}^d$ *be a function with the property that for all scalar test functions* $\widetilde{\phi} \in C_c^\infty\big(\Omega\big)$ *it holds that*

$$\int_\Omega u \cdot \nabla\widetilde{\phi}\, dx = 0.$$

Then we say that u is divergence-free in the sense of distributions*.*

Throughout this section, we will adopt the terminology that the divergence of a vector field is always taken in the sense of distributions.

As an alternative to Definition 2.5, we also have the following definition of distributional solutions to the incompressible Navier-Stokes equation (2.1a):

Definition 2.7 *Let* $u\colon [0,T)\times\Omega \to \mathbb{R}^d$ *be a function with the property that for all divergence-free test functions* $\phi \in C_c^\infty\big((0,\infty)\times\Omega;\mathbb{R}^d\big)$ *it holds that*

$$\int_0^\infty \int_\Omega u\cdot\partial_t\phi + \mu\nabla\phi\cdot\nabla u + \big((u\cdot\nabla)\phi\big)\cdot u\, dxdt = 0, \tag{2.9}$$

and with the property that

$$\lim_{t\to 0^+} \|u(t,x) - u_0(x)\|_{L^2} = 0.$$

Then we say that u solves the incompressible Navier-Stokes equation (2.1a) *in the sense of distributions.*

We remark that the analysis of such weak solutions to Equation (2.1a) presents two main difficulties: the non-linear nature of the convective term and the divergence constraint on the velocity field.

2.2.1 Hodge Decomposition and Leray Projector

In this section, we will attempt to characterise a class of divergence-free functions. We begin by defining some notation.

Let $d \in \{2,3\}$, let $\Omega \subseteq \mathbb{R}^d$ and let ν denote the unit outward normal vector on the boundary $\partial\Omega$. We denote by $L^2_\sigma(\mathbb{R}^d)$ the set given by

$$L^2_\sigma(\mathbb{R}^d) = \big\{u \in L^2(\mathbb{R}^d;\mathbb{R}^d)\colon\ \operatorname{div} u = 0\big\},$$

we denote by $L^2_\sigma(\mathbb{T}^d)$ the set given by

$$L^2_\sigma(\mathbb{T}^d) = \big\{u \in L^2(\mathbb{R}^d;\mathbb{R}^d)\colon\ \operatorname{div} u = 0, \int_{\mathbb{T}^d} u\, dx = 0\big\},$$

and we denote by $L^2_\sigma(\Omega)$ the set given by

$$L^2_\sigma(\Omega) = \big\{ u \in L^2(\mathbb{R}^d;\mathbb{R}^d) \colon u|_{\mathbb{R}^d\setminus\Omega} = 0, (\operatorname{div} u)|_\Omega = 0, (u\cdot\nu)|_{\partial\Omega} = 0 \big\},$$

Theorem 2.8 (Helmholtz-Hodge Decomposition) *Let $d \in \{2,3\}$, let $\Omega \subseteq \mathbb{R}^d$ be either an open, bounded, simply connected, Lipschitz domain, the Torus $\mathbb{T}^d$ or the entire space $\mathbb{R}^d$. Then the space of vector-valued, square integrable functions $L^2(\Omega;\mathbb{R}^d)$ can be written as a direct sum. Indeed, if $d = 2$ it holds that*

$$L^2(\Omega;\mathbb{R}^d) = \big\{\nabla\phi \colon \phi \in H^1(\Omega)\big\} \oplus \big\{(\nabla^\perp\psi) \colon \psi \in L^2(\Omega), (\nabla^\perp\psi) \in L^2(\Omega;\mathbb{R}^d)\big\},$$

and if $d = 3$ it holds that

$$L^2(\Omega;\mathbb{R}^d) = \big\{\nabla\phi \colon \phi \in H^1(\Omega)\big\} \oplus \big\{(\operatorname{curl}\psi) \colon \psi, (\operatorname{curl}\psi) \in L^2(\Omega;\mathbb{R}^d)\big\}.$$

In particular, let $f \in L^2(\Omega;\mathbb{R}^d)$ be any square-integrable vector field. Then there exists a scalar field $g \in H^1(\Omega)$ and a divergence-free vector field $h \in L^2(\Omega;\mathbb{R}^d)$ such that

$$f = \nabla g + h,$$

and

$$\|f\|^2_{L^2(\Omega;\mathbb{R}^d)} = \|\nabla g\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|h\|^2_{L^2(\Omega;\mathbb{R}^d)}.$$

Proof We restrict ourselves to the case of a bounded domain Ω with $d = 3$. The proof for the case $\Omega = \mathbb{R}^d$ and $\Omega = \mathbb{T}^d$ or $d = 2$ is similar.

Let $f \in L^2(\Omega;\mathbb{R}^d)$ and let ν denote the unit outward normal vector at the domain boundary $\partial\Omega$. Consider the boundary value problem given by

$$\begin{aligned} \Delta g &= \overbrace{\operatorname{div} f}^{\in H^{-1}(\Omega)} \quad &&\text{on } \Omega, \\ \frac{\partial g}{\partial \nu} &= \underbrace{f\cdot n}_{\in H^{-1/2}(\Omega)} \quad &&\text{on } \partial\Omega. \end{aligned} \tag{2.10}$$

Equation (2.10) is a Poisson equation with Neumann-type boundary conditions. It can be shown using the theory of elliptic partial differential equations (see, e.g., [Eva10, Chapter 6]) that this equation is well-posed and has, up to an additive constant, a unique solution $g \in H^1(\Omega)$.

Next, we define the vector field $h \in L^2(\Omega;\mathbb{R}^d)$ as

$$h := f - \nabla g.$$

Then, clearly

$$\operatorname{div} h = \operatorname{div} f - \Delta g = 0,$$

and so h is divergence-free. Note also that

$$h \cdot \nu = f \cdot \nu - \frac{\partial g}{\partial \nu} = 0,$$

and therefore $h \in L^2_\sigma(\Omega)$.

Next, observe that the orthogonality of ∇g and h in the L^2 sense,

$$\int_\Omega h \cdot \nabla g dx = -\int_\Omega g \operatorname{div} h \, dx = 0, \tag{2.11}$$

also implies the norm equivalence.

We continue to prove the uniqueness of this decomposition. Let $f_1, f_2 \in L^2_\sigma(\Omega)$ and let $\phi_1, \phi_2 \in H^1(\Omega)$ be functions with the property that

$$\begin{aligned} \operatorname{div} f_1 = \operatorname{div} f_2 = 0 &\quad \text{in } \Omega, \\ f_1 \cdot \nu = f_2 \cdot \nu = 0 &\quad \text{on } \partial\Omega, \end{aligned}$$

and with the property that

$$f = f_1 + \nabla \phi_1 = f_2 + \nabla \phi_2.$$

Then, it holds that

$$f_1 - f_2 = \nabla(\phi_2 - \phi_1)$$

Multiplying both sides of the equation by $f_1 - f_2$ and integrating over the domain Ω, we have

$$\begin{aligned} \int_\Omega |f_1 - f_2|^2 dx &= \int_\Omega (f_1 - f_2) \cdot \nabla(\phi_2 - \phi_1) dx \\ &\underbrace{=}_{\substack{\text{Integration} \\ \text{by parts}}} -\int_\Omega \underbrace{\operatorname{div}(f_1 - f_2)}_{=0.} (\phi_2 - \phi_1) dx = 0. \end{aligned}$$

It follows that $f_1 = f_2$ almost-surely and therefore $\phi_1 = \phi_2$ up to an additive constant.

It remains to show that there exists a unique vector field $\psi \in L^2(\Omega; \mathbb{R}^d)$ such that $h = \operatorname{curl} \psi$. In other words, it remains to show that h is a vector potential. This follows from the Poincaré Lemma (see, e.g., [Spi65, Pg. 94-96]). Incidentally, in a more general sense, this is a consequence of the fact that the de Rham cohomology group of Ω is trivial in the second dimension. The proof is thus complete. □

We are now ready to define the so-called Leray Projector. For the sake of simplicity, unless stated otherwise, we focus on bounded domains Ω but the following definitions can also be extended to domains $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$.

Definition 2.9 (Leray Projector) *Let $d \in \{2,3\}$, let $\Omega \subseteq \mathbb{R}^d$ be a simply connected Lipschitz domain and let $\mathbb{P}\colon L^2(\Omega;\mathbb{R}^d) \to L^2_\sigma(\Omega)$ be an orthonormal projection with the property that for all $f \in L^2(\Omega;\mathbb{R}^d)$ it holds that*

$$\mathbb{P}f = \mathbb{P}(\nabla g + h) = h,$$

where $f = \nabla g + h$ is the Helmholtz-Hodge decomposition of the function f.

Then we call $\mathbb{P}$ the Leray Projector.

Note that Theorem 2.8 clearly indicates that the Leray Projector is well defined.

Remark 2.10 *Consider the setting of Definition 2.9 and let the function $f \in L^2(\Omega;\mathbb{R}^d)$ have the Helmholtz-Hodge decomposition given by $f = \nabla g + h$ so that $\mathbb{P}f = h$. It then follows that*

$$\begin{aligned}\|f\|^2_{L^2(\Omega;\mathbb{R}^d)} &= \|\nabla g\|^2_{L^2(\Omega;\mathbb{R}^d)} + \|h\|^2_{L^2(\Omega;\mathbb{R}^d)} \\ \Longrightarrow \|\mathbb{P}f\|^2_{L^2(\Omega;\mathbb{R}^d)} &= \|h\|^2_{L^2(\Omega;\mathbb{R}^d)} \leq \|f\|^2_{L^2(\Omega;\mathbb{R}^d)}.\end{aligned}$$

One can also show the following Sobolev bound:

$$\|\mathbb{P}f\|^2_{W^{k,2}(\Omega;\mathbb{R}^d)} \leq \|f\|^2_{W^{k,2}(\Omega;\mathbb{R}^d)}.$$

Furthermore, if the vector field f is divergence-free, then clearly $g \equiv C$, where $C \in \mathbb{R}$ is some constant and therefore $\mathbb{P}f = f$, and the previous inequalities become equalities.

Remark 2.11 *Consider the setting of Definition 2.9, let $\Omega = \mathbb{R}^d$ and let the function $f \in L^2(\Omega;\mathbb{R}^d)$ have the Helmholtz-Hodge decomposition given by $f = \nabla g + h$. Then we recall that the function g satisfies the Poisson equation*

$$\Delta g = \operatorname{div} f.$$

It therefore holds that

$$g = -\mathrm{G} * \operatorname{div} f,$$

where G is the fundamental solution of the Laplacian. It follows that

$$\begin{aligned} f =& \mathbb{P}f + \nabla g = \mathbb{P}f - \nabla \mathrm{G} * (\operatorname{div} f) \\ \Longrightarrow& \mathbb{P}f = f + \nabla \mathrm{G} * (\operatorname{div} f) \\ \Longrightarrow& \mathbb{P} = \mathbb{I} + \nabla(-\Delta)^{-1} \operatorname{div},\end{aligned}$$

where the last equation must be understood in the sense of pseudo-differential operators. Denoting by $\widehat{\cdot}$, *the Fourier transform, we obtain*

$$\widehat{\left(\mathbb{P}\phi\right)}_k(\xi) = \widehat{\phi}_k(\xi) - \sum_{j=1}^{d} \frac{\xi_j \xi_k}{|\xi|^2} \widehat{\phi}_j(\xi)$$

Based on the concepts developed thus far, we define notation for some additional function spaces.

Definition 2.12 *Let* $d \in \{2,3\}$ *and let* $\Omega \subseteq \mathbb{R}^d$ *be a simply connected Lipschitz domain. Then we denote by* $\mathcal{V}$ *the set defined as*

$$\mathcal{V} = \{u \in C_c^\infty(\Omega; \mathbb{R}^d) \colon \operatorname{div} u = 0, \}.$$

Definition 2.13 *Let* $d \in \{2,3\}$ *and let* $\Omega \subseteq \mathbb{R}^d$ *be a simply connected Lipschitz domain. Then we denote by* H *the set defined as*

$$H = \overline{\mathcal{V}} \textit{ in } L^2(\Omega; \mathbb{R}^d).$$

It can be shown (see, e.g., [Tem01, Chapter 1, Theorem 1.4]) that $H = L^2_\sigma(\Omega)$.

Definition 2.14 *Let* $d \in \{2,3\}$ *and let* $\Omega \subseteq \mathbb{R}^d$ *be a simply connected Lipschitz domain. Then we denote by* V *the set defined as*

$$V = \overline{\mathcal{V}} \textit{ in } H_0^1(\Omega; \mathbb{R}^d).$$

Once again, it can be shown (see, e.g., [Tem01, Chapter 1, Theorem 1.6]) that $V = H_0^1(\Omega; \mathbb{R}^d) \cap L^2_\sigma(\Omega)$.

Finally, we conclude by noting that using the Leray Projector $\mathbb{P}$, we can rewrite Equation (2.1a) as

$$u_t + \mathbb{P}(u \cdot \nabla)u = \mu \mathbb{P} \Delta u.$$

2.2.2 The Stokes Equations

Let $d \in \{2,3\}$, let $\mu > 0$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with twice-continuously differentiable boundary $\partial\Omega$, let $f \in L^2(\Omega; \mathbb{R}^d)$, let $p \colon \Omega \to \mathbb{R}$ and let $u \colon \Omega \to \mathbb{R}^d$. Then the so-called *Stokes equations* (also known as the

Stationary Navier-Stokes equations) with no-slip boundary conditions are given by

$$\begin{aligned} -\mu\Delta u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.12}$$

Equation (2.12) can be reformulated as a variational problem. To this end, assume that the functions u, p are smooth, let $a := a(\cdot,\cdot)\colon V \times V \to \mathbb{R}$ be a bilinear form with the property that for all $u, v \in V$ it holds that

$$a(u,v) = \mu \int_\Omega \nabla u : \nabla v \, dx,$$

and let $\ell\colon V \to \mathbb{R}$ be the bounded, linear functional defined for all $v \in V$ by

$$\ell(v) := (f, v) = \int_\Omega f \cdot v \, dx.$$

Then, multiplying Equation (2.12) with a test function $v \in \mathcal{V}$ and using integration by parts, we obtain

$$a(u,v) = (f,v). \tag{2.13}$$

Observe that each side of Equation (2.13) depends linearly and continuously on $v \in \mathcal{V}$. Therefore, by continuity, Equation (2.13) also holds for all test functions $v \in V$ where V is the closure of the space $\mathcal{V}$ in $H_0^1(\Omega; \mathbb{R}^d)$.

Furthermore, since u is smooth by assumption and $\partial\Omega$ is C^2-smooth, $u \in H_0^1(\Omega; \mathbb{R}^d)$. Using the fact that u is divergence-free, we also have that $u \in V$. We therefore have the following variational formulation of the Stokes equation (2.12):

Find $u \in V$ such that for all $v \in V$ it holds that

$$a(u,v) = (f,v). \tag{2.14}$$

The next proposition explores the connection between Equation (2.12) and the weak formulation (2.14).

Proposition 2.15 *Let $d \in \{2,3\}$, let $\mu > 0$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with C^2 boundary $\partial\Omega$ and let $f \in L^2(\Omega; \mathbb{R}^d)$. Then, the following are equivalent:*

(i) *There exists* $u \in V$ *that satisfies the variational formulation* (2.14)*.*

(ii) *There exists* $u \in H_0^1(\Omega; \mathbb{R}^d)$ *that satisfies Equation* (2.12) *in the following weak sense:*

There exists a scalar field $p \in L^2(\Omega)$ *such that*

$$\begin{aligned} -\mu\Delta u + \nabla p &= f && \textit{in } \Omega, \\ \operatorname{div} u &= 0 && \textit{in } \Omega, \\ u &= 0 && \textit{on } \partial\Omega, \end{aligned}$$

in the sense of distributions.

(iii) *There exists* $u \in V$ *with the property that*

$$u = \inf_{v \in V} \Phi(v) := \inf_{v \in V} \big(a(v,v) - 2(f,v)\big).$$

Proof The discussion preceding the proposition shows that $(ii) \implies (i)$.

We next prove that $(i) \implies (ii)$. Assume therefore that (i) holds. This immediately implies that $\operatorname{div} u = 0$ in Ω and furthermore that $u|_{\partial\Omega} = 0$ in the sense of traces. Moreover, it holds that

$$-\mu\Delta u - f \in H^{-1}(\Omega; \mathbb{R}^d),$$

and therefore for all $v \in \mathcal{V}$ it holds that

$$\big\langle -\mu\Delta u - f, v\big\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing in the space $\mathcal{V}$.

Finally, applying Proposition A.19 to the function $-\mu\Delta u - f$, we obtain the existence of a scalar field $p \in L^2(\Omega)$ with the property that

$$-\mu\Delta u - f = -\nabla p.$$

Thus, (ii) holds.

We now prove that $(iii) \implies (i)$. Note that for every $v \in V$ and every $\lambda \in \mathbb{R}$ it holds that

$$\begin{aligned} 0 \le \Phi(u + \lambda v) - \Phi(u) &= a(\lambda v, \lambda v) + 2\lambda\big(a(u,v) - (v,f)\big) \\ &= \lambda^2 a(v,v) + 2\lambda\big(a(u,v) - (v,f)\big). \end{aligned}$$

Clearly, $\lambda^2 a(v,v) \ge 0$. Therefore, the above inequality holds for every $v \in V$ and for every $\lambda \in \mathbb{R}$ if and only if

$$a(u,v) - (f,v) = 0.$$

This shows that $(iii) \implies (i)$. Moreover, since the above steps are reversible, the converse is also true, which proves $(i) \implies (iii)$. The proof is complete. □

It now remains to discuss the existence of a unique weak solution to the Stokes equation (2.12). In view of Proposition 2.15 it suffices to prove existence of a unique solution to the variational problem (2.14). This is a simple consequence of the Lax-Milgram Lemma A.20. Indeed, we have the following existence and uniqueness result:

Theorem 2.16 (Weak Solution to Stokes Equation) *There exists a unique weak solution to the Stokes Equation* (2.12) *with no-slip boundary conditions.*

Proof The Poincaré Inequality A.2 implies that the continuous bilinear form $a\colon V \times V \to \mathbb{R}$ is coercive. The Lax-Milgram lemma A.20 therefore implies that there exists a unique solution $u \in V$ to the variational problem (2.14). Proposition 2.15 then implies the existence of a weak solution to the Stokes Equation (2.12). □

We conclude this subsection by defining the so-called *Stokes operator*.

Definition 2.17 (Stokes operator) *Let $d \in \{2,3\}$, let $\Omega \subseteq \mathbb{R}^d$ be a simply connected set with C^2 boundary $\partial\Omega$, let $\mathcal{D}(\mathbb{A})$ denote the set $H^2(\Omega;\mathbb{R}^d) \cap V$, let $\mathbb{P}$ denote the Leray Projector and let $\mathbb{A}\colon \mathcal{D}(\mathbb{A}) \subseteq H \to H$ be the operator defined as*

$$\mathbb{A} := -\mathbb{P}\Delta$$

Then we call $\mathbb{A}$ the Stokes operator.

Remark 2.18 *The motivation for defining the Stokes operator comes from the following observation. Consider the Stokes equation* (2.12) *given by*

$$\begin{aligned} -\mu\Delta u + \nabla p &= f \qquad && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Applying the Leray Projector (*c.f. Definition 2.9*) *to the first equation then implies*

$$\begin{aligned} &-\mu\mathbb{P}\Delta u = \mathbb{P}f \\ \iff &u = \frac{1}{\mu}(-\mathbb{P}\Delta)^{-1}\mathbb{P}f. \end{aligned}$$

Therefore, the Stokes operator allows us to find a solution to the Stokes equations.

We now discuss some properties of the Stokes operator.

Proposition 2.19 *The Stokes operator is*

- *symmetric with respect to the L^2 inner product, that is, for all $u, v \in \mathcal{D}(\mathbb{A})$,*
$$(\mathbb{A}u, v) = (u, \mathbb{A}v),$$
- *self-adjoint, that is, $\mathbb{A} = \mathbb{A}^*$ and $\mathcal{D}(\mathbb{A}) = \mathcal{D}(\mathbb{A}^*)$.*

Proof Recall that by the orthogonality of the Helmholtz-Hodge decomposition, (2.11),
$$\int_\Omega \mathbb{P}u \cdot v\, dx = \int_\Omega u \cdot \mathbb{P}v\, dx,$$
for any $u, v \in H$.

(i) Symmetry: Assume first that $u, v \in \mathcal{V}$ (that is, they are smooth, vanish on the boundary and are divergence-free). Thus, $\mathbb{P}u = u$ and $\mathbb{P}v = v$ and we compute, using integration by parts,

$$\begin{aligned}
(\mathbb{A}u, v) &= -\int_\Omega \mathbb{P}\Delta u \cdot v\, dx \\
&= -\int_\Omega \Delta u \cdot \mathbb{P}v\, dx \\
&= -\int_\Omega \Delta u \cdot v\, dx \\
&= \quad \int_\Omega \nabla u : \nabla v\, dx \\
&= -\int_\Omega u \cdot \Delta v\, dx \\
&= -\int_\Omega \mathbb{P}u \cdot \Delta v\, dx \\
&= -\int_\Omega u \cdot \mathbb{P}\Delta v\, dx = (u, \mathbb{A}v).
\end{aligned}$$

In particular
$$(\mathbb{A}u, v) = \int_\Omega \nabla u : \nabla v\, dx. \tag{2.15}$$

If $u, v \in \mathcal{D}(\mathbb{A})$ are arbitrary, we can approximate them in $H^1(\Omega)$ by functions in $\mathcal{V}$. Passing to the limit in the approximations, we see that (2.15) holds for all $u, v \in \mathcal{D}(\mathbb{A})$. As the right hand side of (2.15) is symmetric in u, v, this proves the first part of the proposition.

(ii) Self-adjointness: Let $u \in \mathcal{D}(\mathbb{A})$. By definition that means that there exists $f \in H$ such that for all $v \in \mathcal{D}(\mathbb{A})$,

$$(\mathbb{A}v, u) = (v, f).$$

Since $f \in L^2(\Omega)$, we can find, by Theorem 2.16, $\widetilde{u}, p$, with $\widetilde{u} \in \mathcal{D}(\mathbb{A})$ such that $\mathbb{A}\widetilde{u} = f$. If we can show that $u = \widetilde{u}$, we are done. To do so, consider for arbitrary $g \in H$ the inner product $(g, u - \widetilde{u})$. Using Theorem 2.16 once more, we can find $v \in \mathcal{D}(\mathbb{A})$ solving $\mathbb{A}v = g$. Hence,

$$(g, u - \widetilde{u}) = (\mathbb{A}v, u) - (\mathbb{A}v, \widetilde{u}) = (v, f) - (v, \mathbb{A}\widetilde{u}) = (v, f) - (v, f) = 0,$$

where we used the symmetry of the Stokes operator for the second equality. □

Theorem 2.20 *The inverse Stokes operator* $\mathbb{A}^{-1} \colon \mathrm{H} \to \mathcal{D}(\mathbb{A})$ *is a bounded, compact, self-adjoint operator.*

Proof See Theorem 2.1 in [Tem01, Chapter 1] and also the comments in [Tem01, Chapter 1, Section 2.6]. □

Theorem 2.20 allows us to apply the spectral theorem for compact self-adjoint operators [Con13, Chapter 2, Theorem 5.1] to the inverse Stokes operator $\mathbb{A}^{-1}$. Indeed, we have the following result:

Theorem 2.21 *Let* $d \in \{2, 3\}$*, let* $\Omega \subseteq \mathbb{R}^d$ *be a simply connected set with* C^2 *boundary* $\partial\Omega$*, let* $\mathbb{A} \colon \mathcal{D}(\mathbb{A}) \subseteq H \to \subseteq H$ *be the Stokes operator and let* $\mathbb{A}^{-1} \colon H \to \mathcal{D}(\mathbb{A})$ *be the inverse Stokes operator. Then there exist positive eigenvalues* $\{\mu_j\}_{j \in \mathbb{N}}$ *of the inverse Stokes operator* $\mathbb{A}^{-1}$ *such that* $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_j \geq \mu_{j+1} \geq \ldots$ *and there exist eigenvectors* $\{w_j\}_{j \in \mathbb{N}}$ *of the inverse Stokes operator* $\mathbb{A}^{-1}$ *such that* $\{w_j\}_{j \in \mathbb{N}}$ *form an orthonormal basis of* H*.*

Proof The proof follows from a simple application of the spectral theorem for compact self-adjoint operators. A detailed argument can be found in Theorem 5.1 and Corollary 5.4 in [Con13, Chapter 2]. □

Theorem 2.21 results in the following important corollary.

Corollary 2.22 *Consider the setting of Theorem 2.21 and for all $j \in \mathbb{N}$, let $\lambda_j = \frac{1}{\mu_j}$. Then,*

(i) for all $j \in \mathbb{N}$ it holds that

$$\mathbb{A} w_j = \lambda_j w_j,$$

(ii) it holds that

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \lambda_{j+1} \leq \ldots,$$

(iii) and we have the limit

$$\lim_{j \to \infty} \lambda_j = \infty.$$

Remark 2.23 *In addition, it can also be shown (see, for example, [Tem01, Chapter 1, Section 2.6]) that if the domain Ω has boundary $\partial\Omega$ of class $C^{\gamma+2}$ for some $\gamma \geq 0$, then the eigenvectors $\{w_j\}_{j\in\mathbb{N}} \subseteq H^{\gamma+2}(\Omega; \mathbb{R}^d)$.*

Finally, we note that if the domain $\Omega = \mathbb{R}^d$ or the torus $\Omega = \mathbb{T}^d$, then the Leray Projector $\mathbb{P}$ and the Laplacian Δ commute.

2.2.3 Leray-Hopf Solutions of the Incompressible Navier-Stokes Equations

We now consider the case of the full incompressible Navier-Stokes equations. We begin by defining and recalling some notation. Throughout this section, let $d \in \{2, 3\}$ and let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded, simply connected domain with smooth boundary $\partial\Omega$. Then we denote by $\mathcal{V}$ the set given by

$$\mathcal{V} = \left\{ \phi \in C_c^\infty\left(\Omega; \mathbb{R}^d\right) : \operatorname{div} \phi = 0 \right\}.$$

We recall that we denote by H the set given by

$$H = L^2_\sigma(\Omega),$$

and we denote by V the set given by

$$V = H^1_0(\Omega; \mathbb{R}^d) \cap L^2_\sigma(\Omega).$$

Definition 2.24 (Leray-Hopf Solutions) *Let $u_0 \in H$ be a vector field and let*

$$u \in L^\infty_{loc}\left([0,\infty), L^2_\sigma(\Omega)\right) \cap L^2_{loc}\left([0,\infty), H^1_0(\Omega;\mathbb{R}^d)\right)$$

be a vector field with the property that u is divergence-free in the sense of distributions (2.8)*, that is, for all $\psi \in C^\infty_c\big(\Omega\big)$ it holds that*

$$\int_\Omega u \cdot \nabla\psi \, dx = 0,$$

and with the property that u solves the incompressible Navier-Stokes equations (2.1a) *in the sense of distributions, specifically, for all divergence-free test functions $\phi \in C^\infty_c((0,\infty)\times\Omega)$ it holds that*

$$\int_0^\infty \int_\Omega u \cdot \partial_t\phi + \mu\nabla\phi\cdot\nabla u + \big((u\cdot\nabla)\phi\big)\cdot u \, dxdt = 0,$$

and additionally it holds that

$$\lim_{t\to 0^+} \|u(t,x) - u_0(x)\|_{L^2} = 0.$$

Then, we say that u is a weak solution of the IBVP (2.2) *and we term u a* Leray-Hopf solution *to the incompressible Navier-Stokes equations.*

We now state the main existence theorem for the incompressible Navier-Stokes equations (2.1).

Theorem 2.25 (Leray-Hopf [Ler34, Hop51]) *Let $d \in \{2,3\}$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with smooth boundary $\partial\Omega$ and let $u_0 \in H$. Then there exists at least one global in time, weak (Leray-Hopf) solution of the IBVP* (2.2) *for the incompressible Navier-Stokes equation with initial datum u_0. Furthermore, the weak solution u satisfies for every $t > 0$ the following energy inequality:*

$$\|u(t)\|^2_{L^2} + 2\mu\int_0^t \|\nabla u(s)\|^2_{L^2} ds \le \|u_0\|^2_{L^2}, \tag{2.16}$$

and in addition it holds that

$$\begin{aligned} d = 2: &\quad \partial_t u \in L^2_{loc}\big([0,\infty); H^{-1}(\Omega)\big), \\ d = 3: &\quad \partial_t u \in L^{4/3}_{loc}\big([0,\infty); H^{-1}(\Omega)\big). \end{aligned} \tag{2.17}$$

Remark 2.26 *In view of Morrey's inequality A.5, Equations* (2.17) *imply weak continuity in time of the Leray-Hopf solution u. Indeed, Morrey's inequality implies that*

$$\begin{aligned} &W^{1,p}(\mathbb{R}) \subset C^{\alpha_p}(\mathbb{R}) \qquad\qquad \textit{where } \alpha_p = \frac{p-1}{p} \\ \implies &u \in C^{\alpha_d}_{loc}\left([0,\infty); H^{-1}(\Omega)\right), \end{aligned}$$

with

$$\alpha_d = \begin{cases} 1/2 & \textit{for } d = 2, \\ 1/4 & \textit{for } d = 3. \end{cases}$$

In order to prove Theorem 2.25, we define the following two bilinear and one trilinear form and prove that they satisfy certain properties.

Definition 2.27 *Let $(\cdot,\cdot)\colon L^2(\Omega;\mathbb{R}^d) \times L^2(\Omega;\mathbb{R}^d) \to \mathbb{R}$ be a bilinear form with the property that for all $u, v \in L^2(\Omega;\mathbb{R}^d)$ it holds that*

$$(u, v) = \int_\Omega u \cdot v \, dx.$$

Definition 2.28 *Let $a := a(\cdot,\cdot)\colon H^1_0(\Omega;\mathbb{R}^d) \times H^1_0(\Omega;\mathbb{R}^d) \to \mathbb{R}$ be a bilinear form with the property that for all $u, v \in H^1_0(\Omega;\mathbb{R}^d)$ it holds that*

$$a(u, v) = \mu \int_\Omega \nabla u : \nabla v \, dx.$$

Definition 2.29 *Let $b := b(\cdot,\cdot,\cdot)\colon V \times V \times V \to \mathbb{R}$ be a trilinear form with the property that for all $u, v, w \in V$ it holds that*

$$b(u, v, w) := \langle \underbrace{B(u, v)}_{\in V^*}, w \rangle = \int_\Omega \big((u \cdot \nabla)v\big) \cdot w \, dx.$$

Note that the trilinear form b is bounded in $V \times V \times V$. Indeed, let $u, v, w \in V$. Then it holds that

$$\begin{aligned} \left| \int_\Omega \big((u \cdot \nabla)v\big) \cdot w \, dx \right| &\underbrace{\leq}_{\text{Hölder's inequality}} \|u\|_{L^4} \|w\|_{L^4} \|\nabla v\|_{L^2} \\ &\underbrace{\leq}_{\text{Ladyzhenskaya's inequality}} \begin{cases} C\|u\|^{1/2}_{L^2} \|\nabla u\|^{1/2}_{L^2} \|w\|^{1/2}_{L^2} \|\nabla w\|^{1/2}_{L^2} \|\nabla v\|_{L^2} & \text{for } d = 2 \\ C\|u\|^{1/4}_{L^2} \|\nabla u\|^{3/4}_{L^2} \|w\|^{1/4}_{L^2} \|\nabla w\|^{3/4}_{L^2} \|\nabla v\|_{L^2} & \text{for } d = 3. \end{cases} \\ &\underbrace{\leq}_{\text{Poincaré inequality}} \tilde{C} \|\nabla u\|_{L^2} \|\nabla w\|_{L^2} \|\nabla v\|_{L^2}. \end{aligned}$$

Therefore, b is a continuous, trilinear operator on $V \times V \times V$.

Next, we state and prove two useful following lemmas regarding the trilinear form b.

Lemma 2.30 *For all $u \in V$ and for all $v \in H_0^1$ it holds that*

$$b(u, v, v) = 0.$$

Proof Assume first $u \in \mathcal{V}$ and $v \in C_c^\infty(\Omega)$. Then we have, by definition,

$$b(u,v,v) = \int_\Omega \big((u \cdot \nabla)v\big) \cdot v \, dx = \frac{1}{2} \int_\Omega u \cdot \nabla |v|^2 dx \underbrace{=}_{\substack{\text{Integration} \\ \text{by parts}}} -\frac{1}{2} \int_\Omega \operatorname{div} u |v|^2 \, dx \underbrace{=}_{u \in V} 0.$$

The general case follows by approximating $u \in V$ and $v \in H_0^1(\Omega)$ by smooth functions in $\mathcal{V}$ and $C_c^\infty(\Omega)$ respectively, and then using the density of $\mathcal{V}$ in V and $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$. $\square$

Lemma 2.31 *For all $u \in V$ and $v, w \in H_0^1(\Omega)$, it holds that*

$$b(u, v, w) = -b(u, w, v).$$

Proof Assume first $u \in \mathcal{V}$ and $v, w \in C_c^\infty(\Omega)$. Then we have, by definition,

$$\begin{aligned} b(u,v,w) &= \int_\Omega \big((u \cdot \nabla)v\big) \cdot w \, dx = \sum_{i,j=1}^d \int_\Omega u_i \partial_i v_j w_j \, dx \\ &\underbrace{=}_{\substack{\text{Integration} \\ \text{by parts}}} - \sum_{i,j=1}^d \int_\Omega (\partial_i u_i v_j w_j + u_i v_j \partial_i w_j) \, dx \\ &= - \int_\Omega \underbrace{\operatorname{div} u}_{=0} (v \cdot w) \, dx - \int_\Omega \big((u \cdot \nabla)w\big) \cdot v \, dx \\ &= -b(u, w, v). \end{aligned}$$

The general case follows by approximating $u \in V$ and $v, w \in H_0^1(\Omega)$ by smooth functions in $\mathcal{V}$ and $C_c^\infty(\Omega)$ respectively, and then using the density of $\mathcal{V}$ in V and $C_c^\infty(\Omega)$ in $H_0^1(\Omega)$. $\square$

Now, using the bilinear and trilinear forms we have just defined, we can state the following variational formulation of the incompressible Navier-Stokes equations (2.1) with initial datum u_0.

For $T > 0$, find $u\colon [0,T] \to V$ such that for all $v \in V$ and for almost every $t \in [0,T]$ it holds that

$$\begin{aligned} \frac{d}{dt}(u(t),v) + b(u(t),u(t),v) + a(u(t),v) &= 0, \\ u(0) = u_0 \quad &\text{in the } L^2 \text{ sense.} \end{aligned} \tag{2.18}$$

Proof (of Theorem 2.25) The following proof is based on the Galerkin method, i.e., by considering finite dimensional subspaces of V and H. An alternative approach is to use mollifications of the involved functions (see e.g., [MB02, Chapter 3]), or use time discretisation after determining solutions to the steady-state problem (see, e.g., [Tem01, Chapter 3]).

We divide the proof in a series of steps.

Step 1

We recall that by Corollary 2.22, there exists an orthonormal basis of the space H consisting of eigenfunctions $\{w_n\}_{n\in\mathbb{N}}$ of the Stokes operator $\mathbb{A}\colon \mathcal{D}(\mathbb{A}) \subseteq H \to\subseteq H$.

We therefore define the space $V_m = \operatorname{span}\{w_1, \dots, w_m\}$ so that for all $m \in \mathbb{N}$ it holds that

$$V_m \subset V_{m+1} \subseteq V.$$

Next, let $T \in (0,\infty)$ and for each $m \in \mathbb{N}$, let $u_m(t) := \sum_{i=1}^m g_{i,m}(t)w_i$ be the function with the property that for all $v \in V_m$ and for a.e. $t \in [0,T]$ it holds that

$$\begin{aligned} \frac{d}{dt}(u_m(t),v) + b(u_m(t),u_m(t),v) + a(u_m(t),v) = 0, \\ u_m(0) = u_{0,m} := \mathbb{P}_m u_0, \end{aligned} \tag{2.19}$$

where $g_{i,m}\colon [0,T] \to \mathbb{R}, i \in \{1,\dots,m\}, m \in \mathbb{N}$ are real-valued functions and $\mathbb{P}_m\colon H \to V_m$ is the orthogonal projection in H on to V_m.

Of course, the existence of such functions $u_m, m \in \mathbb{N}$ for all $t \in [0,T]$ is not a priori clear. In order to prove that such functions do indeed exist, let $m \in \mathbb{N}$, let $v = w_j$ for some $j \in \{1,\dots,m\}$ and expand Equation (2.19) in terms of the basis functions $\{w_i\}_{i=1}^m$:

$$\begin{aligned}&\frac{d}{dt}(u_m(t),v)+b(u_m(t),u_m(t),v)+a(u_m(t),v)=0\\ \implies &\frac{d}{dt}\sum_{i=1}^m g_{i,m}(t)(w_i,w_j)+\sum_{i,k=1}^m g_{i,m}(t)g_{k,m}(t)b(w_i,w_k,w_j)\\ &\qquad\qquad\qquad+\sum_{i=1}^m g_{i,m}(t)a(w_i,w_j)=0. \qquad (2.20)\end{aligned}$$

Observe that by the orthogonality of the eigenvectors $\{w_n\}_{n\in\mathbb{N}}$, for all $i,j\in\{1,\ldots,m\}$ it holds that

$$(w_i,w_j)=\delta_{ij}\quad\text{and}\quad a(w_i,w_j)=\lambda_i\delta_{ij}.$$

Next, for all $i,j\in\{1,\ldots,m\}$ let $\beta_{ikj}:=b(w_i,w_k,w_j)$. Equation (2.20) then reduces to

$$g'_{j,m}(t)+\sum_{i,k=1}^m g_{i,m}(t)g_{k,m}(t)\beta_{ikj}+g_{j,m}(t)\lambda_j=0. \qquad (2.21)$$

Equation (2.21) is a system of non-linear, locally Lipschitz differential equations for the functions $g_{i,m}$, $i\in\{1,\ldots,m\}$. We can supplement this system of ODEs with initial conditions by setting for all $i\in\{1,\ldots,m\}$

$$g_{i,m}(0)=u_{0,m}^{(i)}$$

where $u_{0,m}^{(i)}$ is the i^{th} component of the initial datum $u_{0,m}$.

Standard existence and uniqueness results for ODEs then imply that there exists some $t_m\in(0,\infty]$ such that this initial value problem has a unique solution on the time interval $[0,t_m)$. This proves the (local) existence of the function u_m.

Remark 2.32 *Thus far, for every $m\in\mathbb{N}$, we have only proved local existence of the solution u_m on the open time interval $(0,t_m)$. For each $m\in\mathbb{N}$, if we can show that the solution does not suffer from* blow-up, *i.e., $\lim_{t\to t_m^-}|u_m(t)|\neq\infty$, then, for all $m\in\mathbb{N}$, we will have existence of solutions u_m on the closed interval $[0,T]$. We will use a priori estimates to show that the solution indeed does not suffer from blow-up.*

Step 2
Let $m \in \mathbb{N}$ and let u_m be defined as above. Consider the system of differential equations (2.19) and for each $t \in [0, t_m)$, let $v = u_m(t)$. For all $t \in (0, t_m)$ it then holds that

$$\left(\frac{\partial}{\partial t} u_m(t), u_m(t)\right) + b\big(u_m(t), u_m(t), u_m(t)\big) + \underbrace{a\big(u_m(t), u_m(t)\big)}_{=\mu \|\nabla u_m(t)\|_{L^2}^2} = 0.$$

Lemma 2.31, which states that the trilinear form b is skew-symmetric implies that for every $t \in [0, t_m)$, we have

$$b\big(u_m(t), u_m(t), u_m(t)\big) = 0.$$

We therefore obtain that for all $t \in [0, t_m)$

$$\frac{1}{2} \frac{d}{dt} \|u_m(t)\|_{L^2}^2 + \mu \|\nabla u_m(t)\|_{L^2}^2 = 0,$$

and integrating in time, we obtain that for all $t \in [0, t_m)$ it holds that

$$\|u_m(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u_m(s)\|_{L^2}^2 \, ds = \|u_{0,m}\|_{L^2}^2 \leq \|u_0\|_{L^2}^2. \tag{2.22}$$

Hence, for all $t \in [0, t_m)$, $\|u_m(t)\|_{L^2}^2$ is bounded by $\|u_0\|_{L^2}^2$. This in turn implies that

$$\lim_{t \to t_m^-} \|u_m(t)\|_{L^2} = K < \infty.$$

Note also that by definition for all $t \in [0, t_m)$ it holds that

$$\|u_m(t)\|_{L^2}^2 = \sum_{i,j=1}^{m} g_{i,m} g_{j,m} (w_i, w_j) = \sum_{i=1}^{m} g_{i,m}^2,$$

This implies that the $g_{i,m}$ stay bounded uniformly in t and hence, solutions to the ODE (2.21) do not suffer from blow-up. Therefore, for any arbitrary $T > 0$ we may set $t_m = T$. It follows from (2.22) that for all $T > 0$,

Hence, solutions to the ODE (2.21) do not suffer from blow-up and therefore, for any arbitrary $T > 0$ we may set $t_m = T$. It follows that for all $T > 0$ it holds that

$$\sup_{s\in[0,T]} \|u_m(s)\|_{L^2}^2 \leq \|u_{0,m}\|_{L^2}^2 \leq \|u_0\|_{L^2}^2, \tag{B1}$$

and therefore the sequence of functions $\{u_m\}_{m\in\mathbb{N}}$ is (uniformly in m) bounded in $L^\infty\big([0,T];H\big)$.

Similarly, we obtain from (2.22) that for all $T > 0$

$$2\mu \int_0^T \|\nabla u_m(s)\|_{L^2}^2 \, ds \leq \|u_0\|_{L^2}^2, \tag{B2}$$

and therefore the sequence of functions $\{u_m\}_{m\in\mathbb{N}}$ is (uniformly in m) bounded in $L^2\big([0,T];V\big)$.

Finally, in view of (B1) and (B2) we can use the Banach-Alaoglu theorem A.10 to obtain the existence of a subsequence $\{u_{m_j}\}_{j\in\mathbb{N}}$ and a function $u \in L^\infty\big([0,T];H\big) \cap L^2\big([0,T];V\big)$ with the property that

$$\begin{aligned} u_{m_j} &\overset{*}{\rightharpoonup} u \quad \text{in } L^\infty\big([0,T];H\big) \quad \text{as } j\to\infty, \\ u_{m_j} &\rightharpoonup u \quad \text{in } L^2\big([0,T];V\big) \quad \text{as } j\to\infty. \end{aligned}$$

Notice that these convergence results hold for any $T > 0$ finite, since our choice of T was independent of the parameters.

Throughout the remainder of this proof, we will for simplicity relabel the subsequence $\{u_{m_j}\}_{j\in\mathbb{N}}$ as $\{u_m\}_{m\in\mathbb{N}}$.

Step 3

Our goal is to show that the limiting function u is a weak solution of Navier-Stokes equation and in particular satisfies (2.9) for all test functions in $C_c^\infty(0,\infty;\mathcal{V})$. We therefore consider the variational formulation (2.19) for u_m and analyze what happens to the various terms when we pass to the limit $m\to\infty$. As we are in the end interested in distributional solutions to the Navier-Stokes equations, we test, instead of testing with a function $v_m \in V_m$, with an arbitrary test function

$$\phi(t) = \eta(t)v(x), \quad \eta \in C_c^\infty((0,\infty)),\ v\in\mathcal{V}. \tag{2.23}$$

Using that $\mathbb{P}_m v \in V_m$, we obtain from the finite dimensional problem (2.19) the following expression:

$$\underbrace{\int_0^\infty \big(\partial_t u_m(t), \phi(t)\big)\,dt}_{(v)} + \underbrace{\int_0^\infty b\big(u_m(t), u_m(t), \phi(t)\big)\,dt}_{(vi)} + \underbrace{\int_0^\infty a\big(u_m(t), \phi(t)\big)\,dt}_{(i)}$$

$$= \underbrace{\int_0^\infty \big(\partial_t u_m(t), \phi(t) - \mathbb{P}_m\phi(t)\big)\,dt}_{(ii)} + \underbrace{\int_0^\infty b\big(u_m(t), u_m(t), \phi(t) - \mathbb{P}_m\phi(t)\big)\,dt}_{(iii)}$$
$$+ \underbrace{\int_0^\infty a\big(u_m(t), \phi(t) - \mathbb{P}_m\phi(t)\big)\,dt}_{(iv)}\,. \tag{2.24}$$

We utilize the weak and weak*-convergence of the sequence $\{u_m\}_{m\in\mathbb{N}}$ to analyze the limits of each of the terms in (2.24) as $m \to \infty$.

(i)
$$\int_0^\infty a\big(u_m(t), \phi(t)\big)\,dt = \mu \int_0^\infty \eta(t) \int_\Omega \nabla u_m(t) : \nabla v(t)\,dxdt$$
$$\overset{m\to\infty}{\to} \mu \int_0^\infty \eta(t) \int_\Omega \nabla u(t) : \nabla v(t)\,dxdt = \int_0^\infty a\big(u(t), \phi(t)\big)\,dt,$$

since $\nabla u_m \rightharpoonup \nabla u$ in $L^2((0,T)\times\Omega)$ for any $T > 0$, and $\eta(t)$ has compact support.

(ii) Due to orthogonality of the Stokes eigenfunctions, it holds that

$$\int_\Omega v \cdot \mathbb{P}_m w\,dx = \int_\Omega \mathbb{P}_m v \cdot w\,dx, \quad v, w \in H. \tag{2.25}$$

Hence,

$$\int_0^\infty \big(\partial_t u_m(t), \phi(t) - \mathbb{P}_m\phi(t)\big)\,dt = \int_0^\infty (\partial_t u_m(t), v - \mathbb{P}_m v)\,\eta(t)\,dt = 0.$$

(iii)
$$\Big|\int_0^\infty b\big(u_m(t), u_m(t), \phi(t) - \mathbb{P}_m\phi(t)\big)\,dt\Big|$$
$$= \Big|\int_0^\infty |\eta(t)| \int_\Omega \big((u_m(t)\cdot\nabla)u_m(t)\big)\cdot\big(v - \mathbb{P}_m v\big)dxdt\Big|$$
$$\overset{\text{Lem. 2.31}}{=} \Big|\int_0^\infty |\eta(t)| \int_\Omega ((u_m(t)\cdot\nabla)\,(v - \mathbb{P}_m v))\cdot u_m(t)\,(v - \mathbb{P}_m v)\,dxdt\Big|$$
$$\underbrace{\leq}_{\substack{\text{Hölder's}\\ \text{inequality}}} \int_{\text{supp}(\eta)} \|u_m(t)\|_{L^4}^2 \|\nabla(\phi(t) - \mathbb{P}_m\phi(t))\|_{L^2}\,dt$$

Let T such that $\mathrm{supp}(\eta) \in (0,T)$. Applying now Ladyzhenskaya's inequality, Theorem A.4, to $\|u_m\|_{L^4(\Omega)}$, and using (B1) and (B2) with Hölder's inequality for $p = 4/d$ and $q = 4/(4-d)$, we obtain that

$$\begin{aligned}
&\Big|\int_0^\infty b\big(u_m(t), u_m(t), \phi(t) - \mathbb{P}_m\phi(t)\big)\,dt\Big| \\
&\qquad \le C\int_0^T \|\nabla u_m\|_{L^2}^{d/2}\|u_m(t)\|_{L^2}^{2-d/2}\|\nabla(\phi - \mathbb{P}_m\phi)\|_{L^2}\,dt \\
&\qquad \underbrace{\le}_{\text{(B1)}} C\int_0^T \|\nabla u_m(t)\|_{L^2}^{d/2}\|\nabla(\phi - \mathbb{P}_m\phi)\|_{L^2}\,dt \\
&\qquad \underbrace{\le}_{\text{Hölder}} C\|\nabla u_m\|_{L^2((0,T)\times\Omega)}^{d/2}\left(\int_0^T \|\nabla(\phi(t) - \mathbb{P}_m\phi(t))\|_{L^2}^{\frac{4}{4-d}}\,dt\right)^{\frac{4-d}{4}} \\
&\qquad \underbrace{\le}_{\text{(B2)}} C\|\nabla(\phi - \mathbb{P}_m\phi)\|_{L^{4/(4-d)}((0,\infty);L^2(\Omega))},
\end{aligned}$$

and since $\{w_m\}_{m\in\mathbb{N}} \subseteq H$ is an orthonormal basis of H and $\phi \in C_c^\infty((0,\infty);\mathcal{V})$, it holds that

$$\int_0^\infty \|\phi(t) - \mathbb{P}_m\phi(t)\|_{H_0^1(\Omega)}^{\frac{4}{4-d}}\,dt \overset{m\to\infty}{\longrightarrow} 0$$

and thus

$$\lim_{m\to\infty}\left|\int_0^\infty b\big(u_m(t), u_m(t), \phi(t) - \mathbb{P}_m\phi(t)\big)\,dt\right| = 0.$$

(*iv*) This term is bounded in a similar way as term (*iii*):

$$\begin{aligned}
\left|\int_0^\infty a\big(u_m(t), \phi(t) - \mathbb{P}_m\phi(t)\big)\,dt\right| &= \mu\int_0^\infty\int_\Omega \nabla u_m : \nabla(\phi - \mathbb{P}_m\phi)\,dxdt \\
&\overset{\text{Hölder}}{\le} \mu\|\nabla u_m\|_{L^2((0,T)\times\Omega)}\|\nabla(\phi - \mathbb{P}_m\phi)\|_{L^2((0,T)\times\Omega)} \\
&\overset{\text{(B2)}}{\le} C\|\nabla(\phi - \mathbb{P}_m\phi)\|_{L^2((0,T)\times\Omega)} \\
&\overset{m\to\infty}{\longrightarrow} 0.
\end{aligned}$$

(*v*) We integrate by parts in time and use that η is compactly supported and $u_m \rightharpoonup u$ in $L^2((0,T)\times\Omega)$ for any $T>0$ when we pass to the limit:

$$\begin{aligned}\int_0^\infty \big(\partial_t u_m(t), \phi(t)\big)\,dt &= -\int_0^\infty \eta'(t)\int_\Omega u_m(t)v(x)\,dx\,dt\\ &\overset{m\to\infty}{\longrightarrow} -\int_0^\infty \eta'(t)\int_\Omega u(t)v(x)\,dx\,dt\\ &= -\int_0^\infty (u(t), \partial_t\phi(t))\,dt.\end{aligned}$$

(vi) If we can show that the term $\int_0^\infty b(u_m,u_m,\phi)dt$ converges to $\int_0^\infty b(u,u,\phi)dt$, we have proven that the limit u satisfies (2.9) for test functions of the form (2.23). Unfortunately, this term is nonlinear and therefore the weak convergence of $\{u_m\}_{m\in\mathbb{N}}$ is not sufficient to guarantee convergence to $\int_0^\infty b(u,u,\phi)dt$ as for instance Example A.14 shows. So we need compactness in a stronger topology to be able to conclude. It turns out that strong convergence in $L^2((0,T)\times\Omega)$ is enough as the following argument shows:

Adding and subtracting, the term (vi) can be rewritten as (we assume that $\eta(t)$ has support in $(0,T)$)

$$\begin{aligned}\int_0^\infty b\big(u_m(t),u_m(t),\phi(t)\big)\,dt &\overset{\text{Lem. 2.31}}{=} -\int_0^\infty b\big(u_m(t),\phi(t),u_m(t)\big)\,dt\\ &= -\int_0^T\int_\Omega \big((u_m(t)\cdot\nabla)\phi(t)\big)\cdot u_m(t)\,dxdt\\ &= -\int_0^T\int_\Omega \big((u_m(t)-u(t)\cdot\nabla)\phi(t)\big)\cdot(u_m(t)-u(t))\,dxdt\\ &\quad -\int_0^T\int_\Omega \big(((u_m(t)-u(t))\cdot\nabla)\phi(t)\big)\cdot u(t)\,dxdt\\ &\quad -\int_0^T\int_\Omega \big((u(t)\cdot\nabla)\phi(t)\big)\cdot(u_m(t)-u(t))\,dxdt\\ &\quad -\int_0^T\int_\Omega \big((u(t)\cdot\nabla)\phi(t)\big)\cdot u(t)\,dxdt.\end{aligned}$$

We consider each term separately. The second and third term on the right hand side converge to zero as $m\to\infty$ due to the weak convergence of $u_m \rightharpoonup u$. The last term is what we would like to have, so let us show that also the first term goes to 0 as $m\to\infty$ if we assume that $u_m \to u$ in $L^2((0,T)\times\Omega)$.

$$\left|\int_0^T \int_\Omega \big((u_m(t) - u(t) \cdot \nabla)\phi(t)\big) \cdot (u_m(t) - u(t))\, dxdt\right|$$

$$\underbrace{\leq}_{\substack{\text{Hölder's}\\ \text{inequality}}} \int_0^T \|u_m(t) - u(t)\|_{L^4}^2 \|\nabla\phi(t)\|_{L^2}\, dt,$$

$$\leq \int_0^T \|u_m(t) - u(t)\|_{L^2}^{2-d/2} \|\nabla(u_m(t) - u(t))\|_{L^2}^{d/2} \|\nabla\phi(t)\|_{L^2}\, dt,$$

where the last inequality follows from Ladyzhenskaya's inequality. We can then use the triangle inequality to obtain

$$\left|\int_0^T \int_\Omega \big((u_m(t) - u(t) \cdot \nabla)\phi(t)\big) \cdot (u_m(t) - u(t))\, dxdt\right|$$

$$\leq \int_0^T \|u_m(t) - u(t)\|_{L^2}^{2-d/2} \big(\|\nabla u_m(t)\|_{L^2} + \|\nabla u(t))\|_{L^2}\big)^{d/2} \|\nabla\phi(t)\|_{L^2}\, dt$$

$$\leq \|u_m - u\|_{L^2((0,T)\times\Omega)}^{2-d/2} \left(\|\nabla u_m\|_{L^2} + \|\nabla u\|_{L^2}\right)^{d/2} \|\nabla\phi\|_{L^\infty(0,T;L^2(\Omega))}$$

$$\overset{\text{(B2)}}{\leq} C\|u_m - u\|_{L^2((0,T)\times\Omega)}^{2-d/2} \|\nabla\phi\|_{L^\infty(0,T;L^2(\Omega))}$$

Therefore, under the assumption that the sequence $u_m \to u$ in the strong $L^2\big([0,T]\times\Omega\big)$ sense, the above term also converges to zero. We can therefore conclude that for all $\phi = \eta v$ as in (2.23), it holds that

$$\lim_{m\to\infty} \int_0^\infty b(u_m(t), u_m(t), \phi(t))\, dt = \int_0^\infty b(u(t), u(t), \phi(t))\, dt.$$

Thus, Equation (2.24) implies that if the sequence $u_m \to u$ in the strong $L^2\big([0,T]\times\Omega\big)$ sense then for it holds that

$$\int_0^T \big(\partial_t u(t), \phi(t)\big) + b\big(u(t), u(t), \phi(t)\big) + a\big(u(t), \phi(t)\big)\, dt = 0,$$

for ϕ of the form (2.23). The general case for $\phi \in C_c^\infty(0,\infty;\mathcal{V})$ follows by standard results on approximation of functions.

Step 4
To show that $u_m \to u$ in $L^2((0,T)\times\Omega)$, we use the Aubin-Lions lemma

A.17. We note that for $X = V$ and $Y = H$, assumption (A1) of the lemma is already satisfied. We now show that (A2) is satisfied for $Z = H^{-1}(\Omega)$, $q = p_d$, where $p_d = {}^4\!/\!{}_d$ for $d = 2, 3$, that is,

$$\partial_t u_m \in L^{p_d}\big(0, T, H^{-1}(\Omega)\big) \quad \text{uniformly in } m \in \mathbb{N},$$

where $p_d = 2$ for $d = 2$ and $p_d = {}^4\!/\!{}_3$ for $d = 3$.

We recall from the finite dimensional variational formulation (2.19) that

$$\int_0^\infty \big(\partial_t u_m(t), \phi(t)\big)\, dt = -\int_0^T b\big(u_m(t), u_m(t), \mathbb{P}_m\phi(t)\big)\, dt - \int_0^T a\big(u_m(t), \mathbb{P}_m\phi(t)\big)\, dt.$$

Since u_m satisfies the conditions (B1) and (B2) for every $m \in \mathbb{N}$, it therefore holds that the two terms on the right side of the equation are bounded in $L^{p_d}\big(0, T, H^{-1}(\Omega)\big)$. Specifically, observe that for all $\phi \in \left(L^{p_d}(0, T; H^{-1}(\Omega))\right)^* = L^{p_d'}(0, T; H_0^1(\Omega))$, where $p_d' = p_d/(p_d - 1)$, it holds that

$$\begin{aligned}\left|\int_0^T a(u_m(t), \mathbb{P}_m\phi(t))dt\right| &= \left|\mu \int_0^T \int_\Omega \nabla u_m : \nabla \mathbb{P}_m\phi\, dxdt\right| \\ &\le \mu \|\nabla u_m\|_{L^2((0,T)\times\Omega)} \|\nabla \mathbb{P}_m\phi\|_{L^2((0,T)\times\Omega)} \\ &\le \mu \|\nabla u_m\|_{L^2((0,T)\times\Omega)} \|\nabla \mathbb{P}\phi\|_{L^2((0,T)\times\Omega)} \\ &\le \mu \|\nabla u_m\|_{L^2((0,T)\times\Omega)} \|\phi\|_{L^2(0,T;H_0^1(\Omega))},\end{aligned}$$

where we have used that the eigenfunctions of the Stokes operator are orthogonal for the second inequality and Remark 2.10 for the third. Using the bound (B2), we then obtain

$$\begin{aligned}\left|\int_0^T a(u_m(t), \mathbb{P}_m\phi(t))dt\right| &\le C\|\phi\|_{L^2(0,T;H_0^1(\Omega))} \\ &\le C\|\phi\|_{L^{p_d'}(0,T;H_0^1(\Omega))}.\end{aligned}$$

We proceed to estimating the other term.

$$\begin{aligned}\left|\int_0^T b\big(u_m(t), u_m(t), \mathbb{P}_m\phi(t)\big)\, dt\right| &= \left|\int_0^T \int_\Omega \left(\left(u_m \cdot \nabla\right) u_m\right) \cdot \mathbb{P}_m\phi\, dxdt\right| \\ &\overset{\text{Lem. 2.31}}{=} \left|\int_0^T \int_\Omega \left(\left(u_m \cdot \nabla \mathbb{P}_m\phi\right)\right) \cdot u_m\, dxdt\right| \\ &\le \int_0^T \|u_m(t)\|_{L^4}^2 \|\nabla \mathbb{P}_m\phi\|_{L^2}\, dt.\end{aligned}$$

Now we apply Ladyzhenskaya's inequality to the last expression to obtain that for $d \in \{2,3\}$

$$\begin{aligned}
&\int_0^T \|u_m(t)\|_{L^4}^2 \|\nabla \mathbb{P}_m \phi\|_{L^2}\,dt \\
&\le C \int_0^T \|u_m(t)\|_{L^2}^{2-d/2} \|\nabla u_m(t)\|_{L^2}^{d/2} \|\nabla \mathbb{P}_m \phi\|_{L^2}\,dt \\
&\le C \|u_m(t)\|_{L^\infty(0,T;L^2(\Omega))}^{2-d/2} \int_0^T \|\nabla u_m(t)\|_{L^2}^{d/2} \|\nabla \mathbb{P}\phi\|_{L^2}\,dt \\
&\overset{(B1)}{\le} C \int_0^T \|\nabla u_m(t)\|_{L^2}^{d/2} \|\nabla \mathbb{P}\phi\|_{L^2}\,dt \\
&\overset{\text{Hölder}}{\le} C \|\nabla u_m(t)\|_{L^2((0,T)\times\Omega)}^{d/2} \|\nabla \mathbb{P}\phi\|_{L^{4/(4-d)}(0,T;L^2(\Omega))} \\
&\overset{(B2)}{\le} C \|\nabla \mathbb{P}\phi\|_{L^{p_d'}(0,T;L^2(\Omega))} \\
&\le C \|\phi\|_{L^{p_d'}(0,T;H_0^1(\Omega))}
\end{aligned}$$

It therefore follows that

$$\partial_t u_m \in L^{p_d}\big(0,T,H^{-1}(\Omega)\big),$$

where $p_d = {}^4\!/_d$. Combining this with (B1) and (B2), the assumptions of the Aubin-Lions lemma A.17 are satisfied and we conclude that $u_m \to u$ in $L^2((0,T)\times\Omega)$ up to a subsequence.

Finally, we note that in view of Remark 2.26 which implies weak continuity of the solution u, for all Lebesgue points it holds that

$$\lim_{m\to\infty} \int_\Omega u_m(t)\phi\,dx = \int_\Omega u(t)\phi\,dx \quad \text{uniformly in } t \in [0,T].$$

Step 5

Note that since $u_m \in V_m \subset V$, we have $\operatorname{div} u_m(t) = 0$ for any m and therefore using the weak continuity, we have for any test function $\varphi \in C_c^\infty(\Omega)$

$$0 = \lim_{m\to\infty} \int_\Omega \nabla\varphi \cdot u_m(t)\,dx = \int_\Omega \nabla\varphi \cdot u(t)\,dx.$$

In particular, the limit function u is divergence free in the sense of distributions.

In order to conclude this proof we must show that the solution u satisfies the energy inequality, and we must show that the solution u attains the initial value in the L^2 sense.

Step 6
First, observe that by the properties of weak convergence, Theorem A.13, we have

$$\int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \leq \liminf_{m\to\infty} \int_0^t \|\nabla u_m(s)\|_{L^2}^2 \, dx,$$

and furthermore for a.e. $t \in [0, T]$,

$$\|u(t)\|_{L^2}^2 \leq \limsup_{m\to\infty} \|u_m(t)\|_{L^2}^2.$$

Since

$$\limsup(a_m + b_m) \geq \limsup a_m + \liminf b_m,$$

we conclude, using Equation (2.22) that for a.e. $t \in [0, T]$ it holds that

$$\|u(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds \leq \|u_0\|_{L^2}^2.$$

Step 7
Finally, in order to prove that the solution u attains the initial value in the L^2 sense, observe that the weak continuity of u implies that

$$\|u_0\|_{L^2}^2 \leq \liminf_{t\to 0^+} \|u(t)\|_{L^2}^2.$$

Moreover, the energy inequality implies that

$$\limsup_{t\to 0^+} \|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2,$$

and therefore

$$\lim_{t\to 0^+} \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2.$$

Thus, it holds that

$$\begin{aligned}\lim_{t\to 0^+} \|u(t) - u_0\|_{L^2}^2 &= \lim_{t\to 0^+} \Big(\underbrace{\|u(t)\|_{L^2}^2}_{\to \|u_0\|_{L^2}^2} + \|u_0\|_{L^2}^2 - 2 \underbrace{\langle u_0, u(t)\rangle_{L^2}}_{\to \|u_0\|_{L^2}^2} \Big) \\ &= 0,\end{aligned}$$

where we have used the weak continuity for the last term.

The proof is complete. □

Exercise 2.33 *It can be shown that Leray-Hopf solutions constructed in this way satisfy the weak formulation* (2.9) *for more general test functions than* $\phi \in C_c^\infty(0, \infty; \mathcal{V})$. *Go through the steps of the proof and try to find out what requirements on the test functions are needed.*

2.2.4 Uniqueness of Leray-Hopf Solutions

Thus far, we have only proven the existence of weak (Leray-Hopf) solutions to the incompressible Navier-Stokes equation. We now discuss the question of uniqueness of such weak solutions. It turns out that in the case of two spatial dimensions, Leray-Hopf solutions can be shown to be unique.

Theorem 2.34 (Uniquness of Leray-Hopf Solutions in 2-D) *Let $d = 2$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with smooth boundary $\partial\Omega$, let $u_0 = v_0 \in H$, and let u, v be two Leray-Hopf solutions to the IBVP* (2.2) *for the incompressible Navier-Stokes equation with initial datum u_0, v_0. Then $u = v$ a.e.*

Proof Let $T > 0$, let $u, v\colon [0,T] \to V$ be two Leray-Hopf solutions to the IBVP (2.2), let $w = u - v$. Then, for all test functions $\phi\colon [0,T] \to V$ and for a.e. $t \in [0,T]$ it holds that

$$\big(\partial_t w(t), \phi(t)\big) + a\big(w(t),\phi(t)\big) + b\big(u(t), w(t), \phi(t)\big) + b\big(w(t), v(t), \phi(t)\big) = 0,$$
$$\lim_{t\to 0^+} \|w(t)\|_{L^2} = \|w_0\|_{L^2} = 0.$$

In particular we pick $\phi = w$ to obtain that for a.e. $t \in [0,T]$ it holds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \mu\|\nabla w(t)\|_{L^2}^2 &= -\int_\Omega \big((w(t)\cdot\nabla)v(t)\big)\cdot w(t)\,dx \\ &\quad - \int_\Omega \big((u(t)\cdot\nabla)w(t)\big)\cdot w(t)\,dx \\ &\underbrace{=}_{\text{Lemma 2.30}} - \int_\Omega \big((w(t)\cdot\nabla)v(t)\big)\cdot w(t)\,dx \\ &\underbrace{=}_{\text{Lemma 2.31}} \int_\Omega \big((w(t)\cdot\nabla)w(t)\big)\cdot v(t)\,dx\end{aligned}$$

It follows from Hölder's inequality that

$$\begin{aligned}\int_\Omega \big((w(t)\cdot\nabla)w(t)\big)\cdot v(t)\,dx &\leq \|w(t)\|_{L^4}\|\nabla w(t)\|_{L^2}\|v(t)\|_{L^4} \\ &\underbrace{\leq}_{\substack{\text{Ladyzh-}\\\text{enskaya's}\\\text{inequality}}} C\|w(t)\|_{L^2}^{1/2}\|\nabla w(t)\|_{L^2}^{1/2}\|\nabla w(t)\|_{L^2}\|v(t)\|_{L^2}^{1/2}\|\nabla v(t)\|_{L^2}^{1/2} \\ &= C\|w(t)\|_{L^2}^{1/2}\|\nabla w(t)\|_{L^2}^{3/2}\|v(t)\|_{L^2}^{1/2}\|\nabla v(t)\|_{L^2}^{1/2}.\end{aligned}$$

Next, we recall that by Young's inequality, for any $a, b \in \mathbb{R}$, $\epsilon > 0$ and Hölder conjugates $p, q > 0$,

$$|ab| \leq \frac{|\epsilon a|^p}{p} + \frac{|b|^q}{q\epsilon^q}.$$

Thus, we pick $p = \frac{4}{3}, q = 4$ and $\epsilon = \left(\frac{2\mu}{3}\right)^{3/4}$, and apply Young's inequality to obtain that

$$\begin{aligned}\left(\|\nabla w(t)\|_{L^2}^{3/2}\right)&\left(C\|w(t)\|_{L^2}^{1/2}\|v(t)\|_{L^2}^{1/2}\|\nabla v(t)\|_{L^2}^{1/2}\right)\\ &\leq \frac{\mu}{2}\|\nabla w(t)\|_{L^2}^2 + \frac{\tilde{C}}{\mu^3}\|\nabla v(t)\|_{L^2}^2\|v(t)\|_{L^2}^2\|w(t)\|_{L^2}^2.\end{aligned}$$

Therefore, for a.e. $t \in [0, T]$,

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \mu\|\nabla w(t)\|_{L^2}^2 \leq \frac{\mu}{2}\|\nabla w(t)\|_{L^2}^2 + \frac{\tilde{C}}{\mu^3}\|\nabla v(t)\|_{L^2}^2\|v(t)\|_{L^2}^2\|w(t)\|_{L^2}^2$$

$$\implies \frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \frac{\mu}{2}\|\nabla w(t)\|_{L^2}^2 \leq \frac{\tilde{C}}{\mu^3}\|\nabla v(t)\|_{L^2}^2\|v(t)\|_{L^2}^2\|w(t)\|_{L^2}^2.$$

Now applying Grönwall's inequality, we obtain that for a.e. $t \in [0, T]$,

$$\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 \exp\left(\frac{\tilde{c}}{\mu^3}\int_0^t \|\nabla v(s)\|_{L^2}^2\|v(s)\|_{L^2}^2\, ds\right),$$

where the integral in the expontential is bounded thanks to the energy inequality. Since $\|w_0\|_{L^2} = 0$, it follows that for a.e. $t \in [0, T]$, $\|w(t)\|_{L^2}$=0, and therefore $u = v$ almost everywhere.

We remark that the last inequality also gives us stability of Leray-Hopf solutions with respect perturbations of the initial condition. ☐

Remark 2.35 *Unfortunately, the above uniqueness proof relies crucially on the Ladyzhenskaya inequality A.4 in two spatial dimensions, and therefore the same approach does not work in the three-dimensional case. The uniqueness of weak solutions to the incompressible Navier-Stokes equations in three spatial dimensions is an open problem. It is nevertheless possible to show that if a so-called strong solution to the incompressible Navier-Stokes equation exists in 3-D, then this strong solution must be unique.*

Definition 2.36 (Strong Solutions) *Let $d = 3$, let $H^1_\sigma = V$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with smooth boundary $\partial\Omega$, let $u_0 \in H$ and let*

$u\colon [0,T] \to V$ be a weak solution to the IBVP (2.2) for initial datum u_0 with the property that

$$\begin{aligned} d = 2: &\quad u_0 \in L^2_\sigma \text{ and } u \in L^\infty_{loc}(0,T;H^1_\sigma) \cap L^2_{loc}(0,T;H^2_\sigma) \\ d = 3: &\quad u_0 \in H^1_\sigma \text{ and } u \in C_w(0,T;H^1_\sigma) \cap L^2(0,T;H^2_\sigma), \end{aligned}$$

where $C_w(0,T;H^1_\sigma)$ is defined as the space of all weakly continuous functions $v\colon [0,T] \to H^1_\sigma$.

Theorem 2.37 (Uniqueness of Strong Solutions) *Let $d \in \{2,3\}$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with smooth boundary $\partial\Omega$, let $u_0 = v_0 \in H^1_\sigma = V$, and let u, v be two Leray-Hopf solutions to the IBVP (2.2) for the incompressible Navier-Stokes equation with initial datum u_0, v_0 respectively. If v is a strong solution, then $u \equiv v$.*

Proof Let $T > 0$, let $u, v\colon [0,T] \to V$ be two Leray-Hopf solutions to the IBVP (2.2) with initial data u_0, v_0 respectively, and in addition, let v be a strong solution. For the case $d = 3$, the Gagliardo-Nirenberg interpolation inequality A.3 implies that for a.e. $t \in [0,T]$ there exists some constant C such that

$$\|v(t)\|_{L^p} \le C \|\nabla v(t)\|^\alpha_{L^r} \|v(t)\|^{1-\alpha}_{L^q},$$

where $1 \le q, r \le \infty$, $\alpha \in [0,1]$ and $p \in \mathbb{R}$ is given by

$$\frac{1}{p} = \left(\frac{1}{r} - \frac{1}{3}\right)\alpha + \frac{1-\alpha}{q}.$$

In particular, for $r = q = 2$ it holds that

$$\begin{aligned} \frac{1}{p} &= \frac{\alpha}{6} + \frac{1-\alpha}{2} \\ \implies p &= \frac{6}{3-2\alpha}, \quad \alpha = \frac{3}{2} - \frac{3}{p}. \end{aligned}$$

Thus, for $\alpha \in [0,1]$ we have $p \in [2,6]$. We can therefore conclude that for a.e. $t \in [0,T]$ and for every $p \in [2,6]$ there exists some constant C such that

$$\|v(t)\|_{L^p} \le C \|\nabla v(t)\|^{\frac{3}{2}-\frac{3}{p}}_{L^2} \|v(t)\|^{\frac{3}{p}-\frac{1}{2}}_{L^2},$$

and in particular for a.e. $t \in [0,T]$ there exists some constant C such that

$$\|v(t)\|_{L^3} \le C \|\nabla v(t)\|^{\frac{1}{2}}_{L^2} \|v(t)\|^{\frac{1}{2}}_{L^2}. \tag{2.26}$$

The remainder of the proof is very similar to the proof of Theorem 2.34. Let $w = u - v$. Then, for all test functions $\phi\colon [0,T] \to V$ and for a.e. $t \in [0,T]$ it holds that

$$\big(\partial_t w(t), \phi(t)\big) + a\big(w(t), \phi(t)\big) + b\big(w(t), v(t), \phi(t)\big) + b\big(u(t), w(t), \phi(t)\big) = 0,$$
$$\lim_{t\to 0^+} \|w(t)\|_{L^2} = \|w_0\|_{L^2} = 0.$$

In particular we pick $\phi = w$ to obtain that for a.e. $t \in [0,T]$ it holds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \mu\|\nabla w(t)\|_{L^2}^2 &= -\int_\Omega \big((w(t)\cdot\nabla)v(t)\big)\cdot w(t)\,dx \\ &\quad - \int_\Omega \big((u(t)\cdot\nabla)w(t)\big)\cdot w(t)\,dx \\ &\underbrace{=}_{\text{Lemma 2.30}} - \int_\Omega \big((w(t)\cdot\nabla)v(t)\big)\cdot w(t)\,dx.\end{aligned}$$

Next, observe that by Hölder's inequality it holds that

$$\int_\Omega \big((w(t)\cdot\nabla)v(t)\big)\cdot w(t)\,dx \le \|w(t)\|_{L^3}^2 \|\nabla v(t)\|_{L^3},$$

and now applying the estimate (2.26) obtained from the Gagliardo-Nirenberg interpolation inequality we obtain that

$$\|w(t)\|_{L^3}^2\|\nabla v(t)\|_{L^3} \le C\|w(t)\|_{L^2}\|\nabla w(t)\|_{L^2}\|\nabla v(t)\|_{L^2}^{1/2}\|\nabla^2 v(t)\|_{L^2}^{1/2}.$$

We pick $p = q = 2$ and $\epsilon = \mu^{1/2}$, and apply Young's inequality to the obtain that

$$\begin{aligned}&\Big(\|\nabla w(t)\|_{L^2}\Big)\Big(C\|w(t)\|_{L^2}\|\nabla v(t)\|_{L^2}^{1/2}\|\nabla^2 v(t)\|_{L^2}^{1/2}\Big) \\ &\qquad\qquad \le \frac{\mu}{2}\|\nabla w(t)\|_{L^2}^2 + \frac{\tilde{C}}{\mu}\|w(t)\|_{L^2}^2\|\nabla v(t)\|_{L^2}^2\|\nabla^2 v(t)\|_{L^2}^2.\end{aligned}$$

Therefore, for a.e. $t \in [0,T]$ it holds that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \mu\|\nabla w(t)\|_{L^2}^2 \le \frac{\mu}{2}\|\nabla w(t)\|_{L^2}^2 + \frac{\tilde{C}}{\mu}\|w(t)\|_{L^2}^2\|\nabla v(t)\|_{L^2}^2\|\nabla^2 v(t)\|_{L^2}^2 \\ &\implies \frac{1}{2}\frac{d}{dt}\|w(t)\|_{L^2}^2 + \frac{\mu}{2}\|\nabla w(t)\|_{L^2}^2 \le \frac{\tilde{C}}{\mu}\|w(t)\|_{L^2}^2 \underbrace{\|\nabla v(t)\|_{L^2}^2\|\nabla^2 v(t)\|_{L^2}^2}_{\text{bounded by hypothesis}},\end{aligned}$$

and applying Grönwall's inequality, we obtain that for a.e. $t \in [0,T]$,

$$\|w(t)\|_{L^2}^2 \leq \|w_0\|_{L^2}^2 \exp\left(\frac{\tilde{c}}{\mu}\int_0^t \|\nabla v(s)\|_{L^2}^2 \|\nabla^2 v(s)\|_{L^2}^2 \, ds\right).$$

Since $\|w_0\|_{L^2} = 0$, this implies that for a.e. $t \in [0,T]$, $\|w(t)\|_{L^2}$=0, and therefore $u \equiv v$. □

We conclude this subsection by stating some additional results on uniqueness of solutions to the incompressible Navier-Stokes equations in three spatial dimensions. In fact, the literature contains additional refinements of Theorem 2.37. We state, without proof, one such result.

Theorem 2.38 *[Lio69, Theorem 6.9, Pg. 84] Let $d \in \{2,3\}$, let $T > 0$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with smooth boundary $\partial\Omega$ and let $u_0 = v_0 \in H^1_\sigma = V$. Then there exists, at most, one solution u to the IBVP* (2.2) *for the incompressible Navier-Stokes equation with initial datum u_0 such that*

$$\begin{aligned} &u \in L^2\big(0,T;V\big) \cap L^\infty\big(0,T;H\big), \\ &u \in L^8\big(0,T;L^4(\Omega)\big). \end{aligned} \tag{2.27}$$

Moreover, such a solution is continuous from $[0,T]$ into H.

Remark 2.39 *Consider the setting of Theorem 2.38. One can also replace the result* (2.27) *with the conclusion that the unique solution $u \in L^p\big(0,T;L^q(\Omega)\big)$ where*

$$\begin{aligned} &\frac{2}{p} + \frac{d}{q} \leq 1 \qquad \text{if } \Omega \text{ is bounded,} \\ &\frac{2}{p} + \frac{d}{q} = 1 \qquad \text{if } \Omega \text{ is unbounded.} \end{aligned}$$

Furthermore, if this solution exists, then it is unique in $L^2\big(0,T;V\big) \cap L^\infty\big(0,T;H\big)$ and in the space of functions with finite L^p-norm in time and finite L^q-norm in space.

Finally, we also have the following interesting result.

Theorem 2.40 *Let $d = 3$, let $T > 0$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with smooth boundary $\partial\Omega$ and let $u_0 \in H^1_\sigma$. Let u be a weak solution to the IBVP* (2.2) *for the incompressible Navier-Stokes equation with initial* *datum u_0 such that u is has finite L^p-norm in time and finite L^q-norm in space where*

$$\frac{2}{p} + \frac{d}{q} < 1.$$

Then, u is in fact a strong solution, i.e., continuously differentiable in space on the time interval $[0, T]$. Under the weaker constraint that the initial condition $u_0 \in L^2_\sigma$, then it holds that u is a strong solution on the time interval $[t_0, T]$ for any $t_0 > 0$.

Proof Theorem 2.40 and its proof can be found in a paper of J. Serrin [Ser62, Section 3]. □

2.2.5 Results on Time-Continuity

In this section, we will explore the time-continuity of functions of time and space. These results will be useful in our study of the time-dependent problem. We begin with a simple lemma.

Lemma 2.41 (Lemma 1.1, p. 250 [Tem01]) *Let X be a Banach space with dual space denoted by X^* and let $u, g \in L^1(a, b; X)$. Then the following are equivalent.*

(i) u is a.e. equal to a primitive function of g, i.e, there exists $\xi \in X$ such that for a.e. $t \in [a, b]$,

$$u(t) = \xi + \int_a^t g(s)dx.$$

(ii) For each test function $\phi \in C_0^\infty\big((a, b)\big)$,

$$\int_a^b u(t)\phi'(t)dt = -\int_a^b g(t)\phi(t)dt.$$

(iii) For each $\eta \in X^*$,

$$\frac{d}{dt}\langle u(t), \eta\rangle = \langle g(t), \eta\rangle \qquad \textit{in the sense of distributions.}$$

If either of the conditions $(i) - (iii)$ is satisfied, then in particular, u is a.e. equal to a continuous function from $[a, b]$ into X.

Proof $(i) \implies (iii)$:
Let $\eta \in X^*$. Then for all $\phi \in C_0^\infty\big((a,b)\big)$,

$$
\begin{aligned}
\frac{d}{dt}\langle u,\eta\rangle \underbrace{:=}_{\substack{\text{distri-}\\\text{butional}\\\text{sense}}} -\int_a^b \langle u,\eta\rangle\phi'(t)\,dt &\overset{\text{(i)}}{=} -\int_a^b \langle \xi,\eta\rangle\phi'(t)\,dt \\
&\quad -\int_a^b \Big\langle \int_a^t g(s)\,ds,\eta\Big\rangle\phi'(t)\,dt \\
&= -\langle \xi,\eta\rangle \underbrace{\big(\phi(b)-\phi(a)\big)}_{=0.} - \int_a^b\int_a^t \langle g(s),\eta\rangle\,ds\,\phi'(t)\,dt \\
\underbrace{=}_{\substack{\text{Integration}\\\text{by parts}}} &\int_a^b \frac{d}{dt}\Big(\int_a^t \langle g(s),\eta\rangle\,ds\Big)\phi(t)\,dt \\
= &\int_a^b \langle g(t),\eta\rangle\,\phi(t)\,dt \underbrace{:=}_{\substack{\text{distri-}\\\text{butional}\\\text{sense}}} \langle g,\eta\rangle
\end{aligned}
$$

$(iii) \implies (ii)$:
For all $\eta \in X^*$ and for all $\phi \in C_0^\infty\big((a,b)\big)$ it holds that

$$
\begin{aligned}
&-\int_a^b \langle u,\eta\rangle\phi'(t)\,dt = \int_a^b \langle g,\eta\rangle\phi(t)\,dt \\
\iff &\Big\langle -\int_a^b u\phi'(t)\,dt - \int_a^b g(t)\phi(t)\,dt,\eta\Big\rangle = 0 \\
\implies &-\int_a^b u\phi'(t)\,dt - \int_a^b g(t)\phi(t)\,dt = 0.
\end{aligned}
$$

$(ii) \implies (i)$:
Let $u_0\colon [a,b] \to X$ be a function with the property that for all $t \in [a,b]$ it holds that

$$
u_0(t) = \int_a^t g(s)\,ds.
$$

Then, u_0 is absolutely continuous and $u_0' = g$. Thus, (i) holds with u replaced by u. Moreover, if we define $v := u - u_0$ then by assumption for all $\phi \in C_0^\infty\big((a,b)\big)$,

$$\int_a^b v(t)\phi'(t)\,dt = 0. \tag{2.28}$$

Therefore, if we can show that $v = \xi$, where $\xi \in X$ is a constant element of X, then we are done. Indeed, for a.e. $t \in [a,b]$ it would imply

$$u(t) = u_0(t) + v(t) = \xi + \int_a^t g(s)\,ds.$$

Hence, let $\phi_0 \in C_0^\infty\big((a,b)\big)$ such that $\int_a^b \phi_0(t)\,dt = 1$. Then for any $\phi \in C_0^\infty\big((a,b)\big)$ it holds that

$$\phi = \lambda\phi_0 + \psi', \tag{2.29}$$

where $\lambda \in \mathbb{R}$ is given by

$$\lambda = \int_a^b \phi(t)\,dt,$$

and $\psi \in C_0^\infty\big((a,b)\big)$ is the function with the property that for all $t \in [a,b]$,

$$\psi(t) = \int_a^t \big(\phi(s) - \lambda\phi_0(s)\big)\,ds.$$

Next, let $\xi := \int_a^b v(s)\phi_0(s)\,ds$. Then for any $\phi \in C_0^\infty\big((a,b)\big)$, we have

$$\begin{aligned}
\int_a^b \big(v(t) - \xi\big)\phi(t)\,dt &= \int_a^b v(t)\phi(t)\,dt - \int_a^b \xi\phi(t)\,dt \\
&= \int_a^b v(t)\phi(t)\,dt - \int_a^b v(s)\phi_0(s)\,ds \underbrace{\int_a^b \phi(t)\,dt}_{=\lambda} \\
&\underbrace{=}_{(2.29)} \int_a^b v(t)\big(\lambda\phi_0(t) + \psi'(t)\big)\,dt - \lambda\int_a^b v(s)\phi_0(s)\,ds \\
&= \lambda\int_a^b v(t)\phi_0(t)\,dt + \underbrace{\int_a^b v(t)\psi'(t)\,dt}_{=0 \text{ by } (2.28)} - \lambda\int_a^b v(s)\phi_0(s)\,ds \\
&= \lambda\xi - \lambda\xi = 0.
\end{aligned}$$

We therefore conclude that for all $\phi \in C_0^\infty\big((a,b)\big)$ it holds that

$$\int_a^b \big(v(t) - \xi\big)\phi(t)\,dt = 0,$$

and therefore $v = \xi$ a.e. The proof is thus complete. ☐

At this point, a natural question to ask would be how Lemma 2.41 is useful in our study of Leray-Hopf solutions. To answer this question, consider the setting of Theorem 2.25 and let $u\colon [0,T] \to V$ be a Leray-Hopf solution. Then, observe that we have previously established that $u, \partial_t u \in L^{4/3}\big(0,T; H^{-1}(\Omega)\big)$ since $u \in L^\infty\big(0,T; L^2(\Omega)\big) \hookrightarrow L^{4/3}\big(0,T; H^{-1}(\Omega)\big)$.

Hence, we can set $X = H^{-1}(\Omega), X^* = H_0^1(\Omega)$ and $u, g := \partial_t u \in L^1\big(0,T;X\big)$. It follows that for all $\eta \in C_0^\infty\big((0,T)\big)$ it holds that

$$-\int_0^T \langle u, \eta\rangle \phi'(t)\,dt = \int_0^T \langle \partial_t u, \eta\rangle \phi(t)\,dt = \int_0^T \langle g, \eta\rangle \phi(t)\,dt.$$

Therefore, condition (iii) of Lemma 2.41 is satisfied. Therefore, conditions $(i), (ii)$ of Lemma 2.41 are also satisfied and the Leray-Hopf solution $u\colon [0,T] \to H^{-1}(\Omega)$ is continuous.

In fact, it is possible to obtain a stronger result than Lemma 2.41.

Lemma 2.42 (Lemma 1.4, p. 263 [Tem01]) *Let $X \subset Y$ be two Banach spaces such that $X \hookrightarrow Y$ is a continuous embedding, let $T > 0$, let $\phi \in L^\infty\big(0,T;X\big)$ be a function that is weakly continuous in Y, i.e., for all $v \in Y^*$ the function given by*

$$t \mapsto \langle \phi(t), v\rangle,$$

is continuous. Then ϕ is also weakly continuous in X.

Proof We can replace the space Y with the closure of the space X in Y, i.e., $\overline{X}^Y$. This allows us to assume that X is dense in Y. Hence, by duality, the dense continuous embedding of X into Y implies that the dual space Y^* is continuously embedded into X^*.

By assumption, for each $\eta \in Y^*$ and for all $t_0 \in [0,T]$, we have

$$\lim_{t\to t_0} \langle \phi(t), \eta\rangle = \langle \phi(t_0), \eta\rangle. \tag{2.30}$$

We must show that Equation (2.30) also holds for all $\eta \in X^*$. To this end, we first show that for all $t \in [0,T]$ it holds that $\phi(t) \in X$ and also

$$\|\phi(t)\|_X \leq \|\phi\|_{L^\infty(0,T;X)}. \tag{2.31}$$

Indeed, by regularising the function $\tilde{\phi}$ such that

$$\tilde{\phi} = \begin{cases} \phi & \text{on } [0,T] \\ 0 & \text{otherwise,} \end{cases}$$

we obtain a sequence of smooth functions $\{\phi_m\}_{m\in\mathbb{N}} \colon [0,T] \to X$ with the property that for all $m \in \mathbb{N}$, for all $t \in [0,T]$ and for all $\eta \in Y^*$ it holds that

$$\|\phi_m(t)\|_X \leq \|\phi\|_{L^\infty(0,T;X)},$$

and

$$\lim_{m\to\infty} \langle \phi_m(t), \eta\rangle = \langle \phi(t), \eta\rangle.$$

Therefore, for all $m \in \mathbb{N}$, for all $t \in [0,T]$ and for all $\eta \in Y^*$ it holds that

$$|\phi_m(t), \eta| \leq \|\phi\|_{L^\infty(0,T;X)} \|\eta\|_{X^*},$$

and we obtain in the limit that for all $t \in [0,T]$ and for all $\eta \in Y^*$ it holds that

$$|\phi(t), \eta| \leq \|\phi\|_{L^\infty(0,T;X)} \|\eta\|_{X^*}$$

Therefore, using the fact that $Y^* \subset X^*$ is dense, we obtain that for all $t \in [0,T]$ it holds that $\phi(t) \in X$ and furthermore, the inequality (2.31) also holds.

Next, let $\eta \in X^*$. Since $Y^* \subset X^*$ is dense, for each $\epsilon > 0$ there exists some $\eta_\epsilon \in Y^*$ such that

$$\|\eta - \eta_\epsilon\|_{X^*} \leq \epsilon.$$

For all $t, t_0 \in [0,T]$ it therefore holds that

$$\begin{aligned} &\langle \phi(t) - \phi(t_0), \eta\rangle = \langle \phi(t) - \phi(t_0), \eta - \eta_\epsilon\rangle + \langle \phi(t) - \phi(t_0), \eta_\epsilon\rangle. \\ \implies &|\langle \phi(t) - \phi(t_0), \eta\rangle| \leq 2\epsilon \|\phi\|_{L^\infty(0,T,X)} + |\langle \phi(t) - \phi(t_0), \eta_\epsilon\rangle|. \end{aligned}$$

Since $\eta_\epsilon \in Y^*$, the weak continuity of ϕ in Y implies that

$$\lim_{t\to t_0} |\langle \phi(t) - \phi(t_0), \eta_\epsilon\rangle| = 0.$$

Hence,

$$\limsup_{t\to t_0} |\langle \phi(t) - \phi(t_0), \eta\rangle| \leq 2\epsilon \|\phi\|_{L^\infty(0,T,X)},$$

and since $\epsilon > 0$ is arbitrary, this proves the claim. □

Remark 2.43 *In particular, if we set $X = H$ and $Y = V^*$ then Lemma 2.42 implies that Leray-Hopf solutions are weakly continuous in H.*

In the case of two spatial dimensions, it is possible to prove an even stronger result.

Lemma 2.44 (Lemma 1.2, Pg. 260 [Tem01]) *Let V, H be Hilbert spaces with the property that $V \subset H = H^* \subset V^*$ and let the function $u \in L^2(0,T;V)$ be such that $\partial_t u \in L^2(0,T;V^*)$. Then u is almost everywhere equal to a continuous function from $[0,T]$ into H and furthermore it holds that*

$$\frac{d}{dt}|u|^2 = 2\langle u', u\rangle \qquad \text{in the sense of distributions on } (0,T). \tag{2.32}$$

Proof We first show that Equation (2.32) holds for $u \in L^2(0,T;V)$ such that $\partial_t u \in L^2(0,T;V^*)$.

To this end, let $\tilde{u}\colon \mathbb{R} \to V$ be the extension of u given by

$$\tilde{u} = \begin{cases} u & \text{on } [0,T] \\ 0 & \text{otherwise,} \end{cases}$$

We can then approximate $\tilde{u}$ by a sequence of smooth function $\{u_m\}_{m\in\mathbb{N}} \subseteq C^\infty(0,T;V)$ with the property that

$$\lim_{m\to\infty} \|u_m - u\|_{L^2_{loc}(0,T;V)} = 0, \tag{2.33}$$

$$\lim_{m\to\infty} \|\partial_t u_m - \partial_t u\|_{L^2_{loc}(0,T;V^*)} = 0. \tag{2.34}$$

Since the sequence $\{u_m\}_{m\in\mathbb{N}}$ is smooth in time, for all $m \in \mathbb{N}$ it holds that

$$\frac{d}{dt}\|u_m(t)\|_H^2 = \frac{d}{dt}\big(u_m(t), u_m(t)\big)_H = 2\big(u_m, u_m'\big) = 2\langle u_m, u_m'\rangle.$$

It follows from Equations (2.33)-(2.34) that

$$|u_m|^2 \overset{m\to\infty}{\to} |u|^2 \qquad \text{in } L^1_{loc}\big((0,T)\big),$$

$$\langle u_m', u_m\rangle \overset{m\to\infty}{\to} \langle u', u\rangle \quad \text{in } L^1_{loc}\big((0,T)\big).$$

Therefore, passing to the limit we obtain that for all $\phi \in C_0^\infty\big((0,T)\big)$ it holds that

$$
\begin{aligned}
&-\int_0^T \|u_m(t)\|_H^2 \phi'(t)\,dt = 2\int_0^T \langle u_m(t), u'(t)\rangle \phi(t)\,dt \\
\implies &-\int_0^T \|u(t)\|_H^2 \phi'(t)\,dt = 2\int_0^T \langle u(t), u'(t)\rangle \phi(t)\,dt \\
\implies &\frac{d}{dt}\|u\|_H^2 = 2\langle u, u'\rangle
\end{aligned}
$$

Since $\langle u, u'\rangle \in L^1\big((0,T)\big)$ the above equation implies that $u \in L^\infty(0,T;H)$. It follows by Lemma 2.41 that u is a continuous function from $[0,T]$ into V^*. Moreover, since $u \in L^\infty(0,T;H)$, it follows by Lemma 2.42 that u is weakly continuous from $[0,T]$ into H, i.e., for all $v \in H$ it holds that the mapping

$$t \mapsto \langle u(t), v\rangle, \tag{2.35}$$

is continuous on $[0,T]$.

Next, note that for all $t, t_0 \in [0,T]$

$$\|u(t) - u(t_0)\|_H^2 = \|u(t)\|_H^2 + \|u(t_0)\|_H^2 - 2\big(u(t), u(t_0)\big)_H$$

Moreover, by Equation (2.32) for all $t, t_0 \in [0,T]$ it holds that

$$|u(t)|^2 = |u(t_0)|^2 + 2\int_{t_0}^t \langle u'(s), u(s)\rangle\,ds \implies \lim_{t\to t_0} |u(t)|^2 = |u(t_0)|^2,$$

and therefore, in view of (2.35) we obtain

$$\lim_{t\to t_0} \|u(t) - u(t_0)\|_H^2 = 2|u(t_0)|^2 - 2|u(t_0)|^2 = 0.$$

The proof is thus complete. □

Remark 2.45 *In the context of Leray-Hopf solutions, we have* $V = H^1_\sigma(\Omega)$, $H = L^2_\sigma(\Omega)$. *Note also that if* $d = 3$, *then* $\partial_t u \in L^{4/3}(0, T; V^*)$ *and therefore we cannot apply Lemma 2.44.*

Remark 2.46 *Lemma 2.44 also holds if we assume that* $u \in L^2(0, T; V) \cap L^\infty(0, T; H)$ *and* $\partial_t u \in L^2(0, T; V^*) \cap L^1(0, T; H)$.

2.2.6 Regularity of Leray-Hopf Solutions in 2D

The previous subsections have demonstrated the existence and uniqueness of Leray-Hopf solutions to the incompressible Navier-Stokes equations in two spatial dimensions. We now discuss the regularity properties of such Leray-Hopf solutions. We begin by stating a result on the regularity of the *time-derivative* of Leray-Hopf solutions.

Theorem 2.47 *Let* $d = 2$, *let* $T > 0$, *let* $\Omega \subseteq \mathbb{R}^d$ *be an open, bounded set with smooth boundary* $\partial\Omega$, *let* $V = H^1_\sigma, H = L^2_\sigma$ *and let* $u_0 \in H^2(\Omega) \cap V$. *Then the unique Leray-Hopf solution to the IBVP* (2.2) *of the incompressible Navier-Stokes equation with initial datum* u_0 *satisfies*

$$\partial_t u \in L^2\big(0, T; V\big) \cap L^\infty\big(0, T; H\big).$$

Remark 2.48 *We remark that by the results of the Section 2.2.5, Theorem 2.47 implies that* $u \in C\big([0, T]; V\big)$.

Proof We consider the Galerkin approximation of the variational formulation of the incompressible Navier Stokes equation (2.19):

$$\frac{d}{dt}(u_m(t), v) + b(u_m(t), u_m(t), v) + a(u_m(t), v) = 0,$$
$$u_m(0) = u_{0,m},$$

where $u_{0,m}$ is defined as the orthogonal projection of $u_0 \in H^2(\Omega) \cap V$ onto the finite-dimensional subspace $H^2(\Omega) \cap V_m$. Then note that $u_{0,m} \to u_0$ in $H^2(\Omega)$ as $m \to \infty$. Moreover, we observe that

$$\|u_{0,m}\|_{H^2} \leq \|u_0\|_{H^2}.$$

Next, we pick $v = w_j$ for each $j \in \{1, \ldots, m\}$ where $\{w_j\}_{j \in \mathbb{N}}$ are the eigenvectors of the Stokes operator $\mathbb{A}$. Thus, for each $j \in \{1, \ldots, m\}$ and for a.e. $t \in [0, T]$ it holds that

$$\frac{d}{dt}(u_m(t), w_j) + b(u_m(t), u_m(t), w_j) + a(u_m(t), w_j) = 0. \tag{2.36}$$

Our aim is to show that for all $m \in \mathbb{N}$ and for a.e. $t \in [0,T]$, the time derivative $\frac{d}{dt}u_m(t)$ remains in a bounded set of $L^2\big(0,T;V\big)\cap L^\infty\big(0,T;H\big)$. This will then imply that for a.e. $t \in [0,T]$, the time derivative $\frac{d}{dt}\lim_{m\to\infty} u_m(t) = \frac{d}{dt}u(t)$ also remains in the same bounded set.

To this end, for each $j \in \{1,\dots,m\}$, we multiply equation (3.1) with $g'_{j,m}(t)$ and sum the resulting equations over all $j \in \{1,\dots,m\}$ to obtain that for a.e. $t \in [0,T]$ it holds that

$$\Big(\partial_t u_m(t), \sum_{j=1}^m g'_{j,m}(t)w_j\Big) + b\Big(u_m(t), u_m(t), \sum_{j=1}^m g'_{j,m}(t)w_j\Big) + a\Big(u_m(t), \sum_{j=1}^m g'_{j,m}(t)w_j\Big).$$

Using the fact that the eigenvectors $\{w_j\}_{j\in\mathbb{N}}$ form an orthonormal basis of $H^2(\Omega)\cap V$, we therefore obtain the following equation which holds pointwise for a.e. $t \in [0,T]$:

$$\|\partial_t u_m(t)\|_{L^2}^2 + b\big(u_m(t), u_m(t), \partial_t u_m(t)\big) + a\big(u_m(t), \partial_t u_m(t)\big) = 0. \tag{2.37}$$

Let us now consider Equation (3.2) at time $t = 0$ first:

$$\begin{aligned}
\|\partial_t u_m(0)\|_{L^2}^2 &= -\, b\big(u_m(0), u_m(0), \partial_t u_m(0)\big) - a\big(u_m(0), \partial_t u_m(0)\big)\\
&= -\, b\big(u_{0,m}, u_{0,m}, \partial_t u_m(0)\big) + \underbrace{\mu\big(\Delta u_{0,m}, \partial_t u_m(0)\big)}_{\text{Integration by parts}}
\end{aligned}$$

$$\underbrace{\le}_{\substack{\text{Hölder's}\\ \text{Inequality}}} \|u_{0,m}\|_{L^4}\|\nabla u_{0,m}\|_{L^4}\|\partial_t u_m(0)\|_{L^2} + \mu\|\Delta u_{0,m}\|_{L^2}\|\partial_t u_m(0)\|_{L^2}$$

$$\underbrace{\le}_{\substack{\text{Ladyzh-}\\ \text{enskaya's}\\ \text{inequality}}} C\|u_{0,m}\|_{L^2}^{1/2}\|u_{0,m}\|_{H_0^1}\|u_{0,m}\|_{H^2}^{1/2}\|\partial_t u_m(0)\|_{L^2} + \mu\|u_{0,m}\|_{H^2}\|\partial_t u_m(0)\|_{L^2}$$

$$\underbrace{\le}_{\substack{\text{Projection}\\ \text{property}}} C\|u_0\|_{L^2}^{1/2}\|u_0\|_{H_0^1}\|u_0\|_{H^2}^{1/2}\|\partial_t u_m(0)\|_{L^2} + \mu\|u_0\|_{H^2}\|\partial_t u_m(0)\|_{L^2}$$

$$\underbrace{\le}_{\substack{\text{Young's}\\ \text{inequality}}} \frac{1}{4}\|\partial_t u_m(0)\|_{L^2}^2 + 4C^2\|u_0\|_{L^2}\|u_0\|_{H1}^2\|u_0\|_{H^2} + \frac{1}{4}\|\partial_t u_m(0)\|_{L^2}^2 + 4C^2\mu^2\|u_0\|_{H^2}^2$$

$$\implies \|\partial_t u_m(0)\|_{L^2}^2 \le 8C^2\|u_0\|_{L^2}\|u_0\|_{H1}^2\|u_0\|_{H^2} + 8C^2\mu^2\|u_0\|_{H^2}^2 \le \tilde{C} < \infty.$$

Hence, it holds that $\partial_t u_m(0) \in H$.

Next, let us once again consider Equation (3.1). We differentiate both sides of this equation with respect to time to obtain that for each $j \in \{1, \dots, m\}$ and for a.e. $t \in [0, T]$ it holds that

$$\big(\partial_{tt}^2 u_m(t), w_j\big) + a\big(\partial_t u_m(t), w_j\big) + b\big(\partial_t u_m(t), u_m(t), w_j\big) \\ + b\big(u_m(t), \partial_t u_m(t), w_j\big) = 0.$$

For each $j \in \{1, \dots, m\}$, we multiply the above equation with $g'_{j,m}(t)$ and sum the resulting equations over all $j \in \{1, \dots, m\}$ to obtain that for a.e. $t \in [0, T]$ it holds that

$$\big(\partial_{tt}^2 u_m(t), \partial_t u_m(t)\big) + a\big(\partial_t u_m(t), \partial_t u_m(t)\big) + b\big(\partial_t u_m(t), u_m(t), \partial_t u_m(t)\big) \\ + \underbrace{b\big(u_m(t), \partial_t u_m(t), \partial_t u_m(t)\big)}_{=0 \text{ by Lemma 2.30}} = 0.$$

Therefore for a.e. $t \in [0, T]$ it holds that

$$\frac{1}{2}\frac{d}{dt}\|\partial_t u_m(t)\|_{L^2}^2 + \mu\|\nabla\partial_t u_m(t)\|_{L^2}^2 = -b\big(\partial_t u_m(t), u_m(t), \partial_t u_m(t)\big). \quad (2.38)$$

It now remains to obtain appropriate estimate the right side of Equation (3.3). To this end, we observe that

$$\begin{aligned} |b\big(\partial_t u_m(t), u_m(t), \partial_t u_m(t)\big)| &\underbrace{\leq}_{\substack{\text{Hölder's} \\ \text{Inequality}}} \|\partial_t u_m(t)\|_{L^4}^2 \|\nabla u_m(t)\|_{L^2} \\ &\underbrace{\leq}_{\substack{\text{Ladyzh-} \\ \text{enskaya's} \\ \text{inequality}}} C\|\partial_t u_m(t)\|_{L^2}\|\partial_t \nabla u_m(t)\|_{L^2}\|\nabla u_m(t)\|_{L^2} \\ &\underbrace{\leq}_{\substack{\text{Young's} \\ \text{inequality}}} \frac{\mu}{2}\|\nabla\partial_t u_m(t)\|_{L^2}^2 + \frac{C^2}{2\mu}\|\partial_t u_m(t)\|_{L^2}^2\|\nabla u_m(t)\|_{L^2}^2, \end{aligned}$$

and therefore, for a.e. $t \in [0, T]$ it holds that

$$\frac{d}{dt}\|\partial_t u_m(t)\|_{L^2}^2 + \mu\|\nabla\partial_t u_m(t)\|_{L^2}^2 \leq \frac{C^2}{\mu}\|\partial_t u_m(t)\|_{L^2}^2\|\nabla u_m(t)\|_{L^2}^2.$$

Finally, we apply Gronwall's Lemma to obtain that for a.e. $t \in [0, T]$ it holds that

$$\|\partial_t u_m(t)\|_{L^2}^2 \leq \|\partial_t u_m(0)\|_{L^2}^2 \exp\Big(\tilde{C}\int_0^t \|\nabla u_m(s)\|_{L^2}^2\, ds\Big).$$

We have previously shown that both terms on the right side of the above equation are bounded. Thus, for a.e. $t \in [0,T]$ it holds that

$$\|\partial_t u_m(t)\|_{L^2}^2 \leq K \quad \text{uniformly in } m.$$

Similarly, for a.e. $t \in [0,T]$ it also holds that

$$\int_0^t \|\nabla \partial_t u_m(t)\|_{L^2}^2 \leq \tilde{K} \quad \text{uniformly in } m.$$

It follows that

$$\partial_t u \in L^2\big(0,T;V\big) \cap L^\infty\big(0,T;H\big). \qquad \square$$

We are now ready to state the main result on the regularity of Leray-Hopf solutions.

Theorem 2.49 *Let $d = 2$, let $T > 0$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with twice-continuously differentiable boundary $\partial\Omega$ such that the exterior cone condition is satisfied at each point in $\partial\Omega$ and let $u_0 \in H^2(\Omega) \cap V$. Then the unique Leray-Hopf solution to the IBVP (2.2) of the incompressible Navier-Stokes equation with initial datum u_0 satisfies*

$$u \in L^\infty\big(0,T;H^2(\Omega)\big).$$

Proof Let $u\colon [0,T] \to \mathbb{R}$ be the unique Leray-Hopf solution to the IBVP (2.2) of the incompressible Navier-Stokes equation with initial datum u_0 and consider the following weak formulation:

$$\big(\partial_t u(t), v\big) + b\big(u(t), u(t), v\big) + a\big(u(t), v\big) \quad \forall v \in V.$$

This weak formulation can be re-written as

$$a\big(u(t), v\big) = \big(g(t), v\big) \quad \forall v \in V, \tag{2.39}$$

where the function $g\colon [0,T] \to V^*$ is the function with the property that for all $t \in [0,T]$ it holds that

$$g(t) = -\partial_t u(t) - \underbrace{B(u(t), u(t))}_{\text{c.f. Def. 2.29}}$$

We have shown in the previous theorem that $u \in L^\infty\big(0,T;V\big)$ is unique. Moreover, for a.e. $t \in [0,T]$ it holds that

$$
\begin{aligned}
|b\big(u(t),u(t),v\big)| &\underbrace{\leq}_{\substack{\text{Hölder's}\\ \text{inequality}}} \|u(t)\|_{L^4}\|u(t)\|_{H_0^1}\|v\|_{L^4} \\
&\underbrace{\leq}_{\substack{\text{Ladyzh-}\\ \text{enskaya's}\\ \text{inequality}}} C \underbrace{\|u(t)\|_{H_0^1}^{3/2}\|\|_{L^2}^{1/2}}_{\in L^\infty\left([0,T]\right)} \|v\|_{L^4}.
\end{aligned}
$$

It follows that the mapping $B(u,u) \in L^{\infty,4/3}_{t,x}$, $\partial_t u \in L^{\infty,2}_{t,x}$, and therefore $g \in L^{\infty,4/3}_{t,x}$. Since the Leray-Hopf solution u exists and is unique, it follows that the function g also exists and is well-defined.

Thus, Equation (3.4) can be viewed as the variational formulation of an elliptic PDE with right-hand side given by $g(t) \in L^{4/3}(\Omega)$ for a.e. $t \in [0,T]$:

For $T > 0$, and a.e. $t \in [0,T]$, find $u(t)$ such that for all $v \in V$ it holds that

$$
\begin{aligned}
a\big(u(t),v\big) &= \big(g(t),v\big) \quad &&\text{on } \Omega,\\
u(t) &= 0 &&\text{on } \partial\Omega.
\end{aligned} \tag{2.40}
$$

We already know that the solution u to the above variational problem is unique and furthermore $u(t) \in H_0^1$ for a.e. $t \in [0,T]$. We can therefore apply Theorem A.21 [GT15, Theorem 8.30, Chapter 8]. Indeed, we observe that the elliptic operator in the case of the variational problem (2.40) is coercive and has bounded coefficients that satisfy the non-positivity condition. Furthermore, $f \equiv 0$ and $g \in L^{4/3}$ with $\frac{4}{3} > \frac{d}{2} = 1$, and moreover the boundary condition $\phi \equiv 0 \in C^0(\partial\Omega)$. Hence, in view of Theorem A.21, for a.e. $t \in [0,T]$ it holds that

$$
u(t) \in C^0(\bar{\Omega}).
$$

Let us now reconsider the trilinear form b. For a.e. $t \in [0,T]$ it holds that

$$
|b\big(u(t,)u(t),v\big)| \leq \|u(t)\|_{L^\infty}\|u(t)\|_{H_0^1}\|v\|_{L^2} \implies \quad B\big(u,u\big) \in L^{\infty,2}_{t,x}.
$$

Thus, the right-hand side of the variational formulation (2.40) satisfies the property that $g(t) \in L^2(\Omega)$ for a.e. $t \in [0,T]$. Standard theory for elliptic

PDEs (see, e.g., [Eva10, Chapter 6.3, Theorem 4]) implies that the solution $u \in H^2(\Omega)$ and furthermore for a.e. $t \in [0,T]$ satisfies the estimate

$$\|u(t)\|_{H^2} \leq C\Big(\|g(t)\|_{L^2} + \|u(t)\|_{L^2}\Big),$$

where $C = C(\Omega)$ is a constant. It therefore follows that $u \in L^\infty\big([0,T]; H^2(\Omega)\big)$.□

Remark 2.50 *In fact, in the case of two spatial dimensions, one can show that the solution $u(t) \in C^\infty(\Omega)$ for a.e. $t \in [0,T]$ if the initial data and the domain boundary $\partial\Omega$ are smooth enough. We refer to the previously mentioned paper of J. Serrin [Ser62].*

Remark 2.51 *On the other hand, in the case of three spatial dimensions, we can either show local existence of strong solutions, or existence of strong solutions for arbitrary times given that the initial data is "small enough".*

The following regularity result elaborates on Remark 2.51

Theorem 2.52 *Let $d = 3$, let $T > 0$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with twice-continuously differentiable boundary $\partial\Omega$, let $u_0 \in H^2(\Omega) \cap V$ and assume that either the kinematic viscosity μ is sufficinetly large or u_0 is "small enough". Then there exists a unique Leray-Hopf solution u to the IBVP* (2.2) *of the incompressible Navier-Stokes equation with initial datum u_0 that satisfies*

$$u \in L^{\infty,4}_{t,x},$$
$$\partial_t u \in L^2\big(0,T;V\big) \cap L^\infty\big(0,T;H\big) \cap L^\infty\big(0,T;H^2(\Omega)\big)$$

Proof See Theorem 3.7 and Theorem 3.8 in [Tem01, Chapter 3]. □

Finally, we also have a result on the *long term behaviour* of solutions to the incompressible Navier-Stokes equations.

Theorem 2.53 *Let $0 < T_2, T_3$, let $d \in \{2,3\}$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded set with twice-continuously differentiable boundary $\partial\Omega$ and let $u_0 \in V$. Then, for $d = 2$ it holds that $u \in L^\infty\big(0,\infty,V\big)$ and u tends to 0 in V as $t \to \infty$, and for $d = 3$ it holds that $u \in L^\infty\big(0,T_2,V\big)$, $u \in L^\infty\big(T_3,\infty,V\big)$ and u tends to 0 in V as $t \to \infty$.*

Proof This theorem and its proof can be found in [Tem01, Theorem 3.12, Pg. 318]. □

2.3 Caffarelli-Kohn-Nirenberg Blow-up Result

In this section, we will present some additional partial regularity results for weak solutions to the incompressible Navier-Stokes equations (2.1). In particular, we will focus on the work of Caffarelli, Kohn and Nirenberg [CKN82] who provided a characterisation of the so-called *blow-up* set associated with weak solutions to the IBVP (2.2)

We begin by recalling from measure theory, the definition of the *Hausdorff measure.*

Definition 2.54 *Let $d \in \mathbb{N}$, let $\alpha \geq 0$ and $\delta > 0$ be constants, let $U \subseteq \mathbb{R}^d$, let ω_α be defined as*

$$\omega_\alpha = \frac{\pi^{\alpha/2}}{\Gamma(1 + \alpha/2)},$$

where Γ is the Gamma function, and let $\mathcal{H}^\alpha_\delta(U)$ be given by

$$\mathcal{H}^\alpha_\delta(U) = \inf \left\{ \sum_i \omega_\alpha \left(\tfrac{\operatorname{diam} F_i}{2}\right)^\alpha : U \subset \bigcup_i F_i \text{ and } \operatorname{diam}(F_i) < \delta \ \forall i \right\}.$$

Then the α-Hausdorff measure of the set U is defined as

$$\mathcal{H}^\alpha(U) := \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(U) = \sup_{\delta > 0} \mathcal{H}^\alpha_\delta(U).$$

Since weak solutions to the IBVP (2.2) are functions of both space and time, we will require a variant of the Hausdorff measure known as the *Parabolic Hausdorff measure.*

For clarity of exposition, we first introduce some notation.

Definition 2.55 *Let $d \in \mathbb{N}$, let $r > 0$, let $x \in \mathbb{R}^d$, let $B_r(x) \subset \mathbb{R}^d$ denote the ball of radius r with centre x, and let $t \in \mathbb{R}$. Then we denote by $Q_r(t,x)$ the set given by*

$$Q_r(t,x) = (t - r^2, t + r^2) \times B_r(x),$$

and we call $Q_r(t,x)$ a parabolic cylinder.

Definition 2.56 *Let $U \subset \mathbb{R} \times \mathbb{R}^3$ and let $\alpha \geq 0$ and $\delta > 0$, let ω_α be defined as*

$$\omega_\alpha = \frac{\pi^{\alpha/2}}{\Gamma(1 + \alpha/2)},$$

where Γ *is the Gamma function, and let* $\mathcal{P}_\delta^\alpha$ *be given by*

$$\mathcal{P}_\delta^\alpha(U) = \inf\left\{ \sum_i \omega_\alpha r_i^\alpha \colon U \subset \bigcup_i Q_{r_i}(t_i, x_i) \text{ and } \underbrace{2r_i\sqrt{1+r_i^2} < \delta}_{\text{equivalently, } r_i < \delta} \ \forall i \right\}.$$

Then the α*-Parabolic Hausdorff measure of the set* U *is defined as*

$$\mathcal{P}^\alpha(U) := \lim_{\delta\to 0} \mathcal{P}_\delta^\alpha(U) = \sup_{\delta>0} \mathcal{P}_\delta^\alpha(U).$$

Some remarks are now in order.

Remark 2.57 *Let* $U \subset \mathbb{R}\times\mathbb{R}^3$*. It can be shown that for every* $\alpha > 0$ *there exists some constant* $C_\alpha > 0$ *such that*

$$\mathcal{H}^{\alpha/2}\Big(\big\{t\colon (\{t\}\times\mathbb{R}^3)\cap U \neq \phi\big\}\Big) \leq C_\alpha \mathcal{P}^\alpha(U).$$

Remark 2.58 *Let* $U \subset \mathbb{R}\times\mathbb{R}^3$*. It can also be shown that for every* $\alpha > 0$ *there exists some constant* $\bar{C}_\alpha > 0$ *such that*

$$\mathcal{H}^{\alpha}\Big(\big\{t\colon \{t\}\times\mathbb{R}^3\cap U\big\}\Big) \leq \bar{C}_\alpha \mathcal{P}^\alpha(U).$$

Remark 2.59 *Let* $U \subset \mathbb{R}\times\mathbb{R}^3$*. Then the set* U *can be covered be a sequence of parabolic cylinders* $\{Q_{r_i}(t_i,x_i)\}_{i=1}^\infty$ *such that for all* $\delta > 0$ *it holds that* $\sum_i r_i^\alpha < \delta$*, if and only if*

$$\mathcal{P}^\alpha(U) = 0.$$

A comprehensive treatment of the properties of the Hausdorff measure can, for example, be found in [Fed14, Section 2.10].

Next, we introduce the notion of so-called *suitable weak solutions* to the incompressible Navier-Stokes equations.

Definition 2.60 (Suitable weak solution) *Let* $J\times\Omega$ *be an open set in* $\mathbb{R}\times\mathbb{R}^3$*, let* $u\colon J\times\Omega\to\mathbb{R}^3$ *be a mapping such that*

- $u \in L^\infty\big(J; L^2(\Omega;\mathbb{R}^3)\big)$ *and also* $\nabla u \in L^2(J\times\Omega;\mathbb{R}^{3\times 3})$*;*
- *there exists some scalar field* $p \in L^{5/3}_{loc}(J\times\Omega)$ *such that the equation*

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla p =& \mu\Delta u, \\ \operatorname{div} u =& 0, \end{aligned}$$

holds for all $(t,x)\in J\times\Omega$ *in the sense of distributions;*

- *and for all non-negative test functions* $\phi \in C_c^\infty(J \times \Omega)$*, it holds that*

$$2\int_{J\times\Omega} |\nabla u|^2 \phi \, dxdt \leq \int_{J\times\Omega} \big(|u|^2(\phi_t + \Delta\phi) + (|u|^2 + 2p)u \cdot \nabla\phi\big) \, dxdt.$$

Then u *is called a local suitable weak solution of the incompressible Navier-Stokes equations* (2.1) *in the domain* $J \times \Omega$.

The existence of suitable weak solutions to the incompressible Navier-Stokes equations can be rigorously proved. Indeed, we have the following existence result for suitable weak solutions:

Theorem 2.61 *Let* $J \times \Omega$ *be an open set in* $\mathbb{R} \times \mathbb{R}^3$ *such that either* $\Omega = \mathbb{R}^3$ *or* $\Omega \subset \mathbb{R}^3$ *is a bounded, open and connected set with smooth boundary* $\partial\Omega$, *let* $\sup J = T$*, and let* $u_0 \colon \Omega \to \mathbb{R}^3$ *be a function with the property that*

$$u_0 \in \begin{cases} H & \text{if } \ \Omega = \mathbb{R}^3 \\ H \cap W^{2/5, 4/5}(\Omega; \mathbb{R}^3) & \text{if } \ \Omega \subset \mathbb{R}^3. \end{cases}$$

Then there exists a suitable weak solution u *of the incompressible Navier-Stokes equations* (2.1) *in the domain* $J \times \Omega$ *and a pressure field* p *such that*

1. $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$;
2. $u(t) \to u_0$ *weakly in* H *as* $t \to 0$;
3. $p \in L^{5/3}(J \times \Omega)$;
4. *and for all non-negative test functions* $\phi \in C^\infty(J \bar{\times} \Omega)$ *such that* $\phi = 0$ *near* $(0,T) \times \partial\Omega$ *and for all* $0 < t < T$ *it holds that*

$$\begin{aligned}\int_\Omega |u|^2(t, x\phi(t,x)) \, dx + 2\int_0^t \int_\Omega |\nabla u|^2(t,x)\phi(t,x) \, dxds &\leq \int_\Omega |u_0(x)|^2 \phi(0,x) \, dx \\ &+ \int_0^t \int_\Omega |u|^2(t,x)\big(\phi_t(t,x) + \Delta\phi(t,x)\big) \, dxds \\ &+ \int_0^t \int_\Omega \big(|u|^2(t,x) + 2p\big) u \cdot \nabla p \, dxds.\end{aligned}$$

Remark 2.62 *When* $\Omega = \mathbb{R}^3$*, the phrase "*$\phi = 0$ *near* $(0,T) \times \partial\Omega$*" is understood to mean that the test function* ϕ *is compactly supported in the space variable.*

Remark 2.63 *Theorem 2.61 also holds in the presence of an additional forcing term f for the incompressible Navier-Stokes equation, provided that $f \in L^2(J \times \Omega)$ and $\operatorname{div} f = 0$. Note that the result for the special case $f = 0$ and $\Omega = \mathbb{R}^3$ was already proved by V. Scheffer in 1977 in [Sch77].*

Remark 2.64 *It is currently not known if suitable weak solutions can be constructed using the Galerkin approximation procedure (cf. Theorem 2.25).*

The complete proof of Theorem 2.61 can be found in [CKN82, Apendix]. We only give the main idea of the proof, which is based on the use of mollifiers. We require the following definition and lemma:

Definition 2.65 *Let $J \times \Omega \subset \mathbb{R} \times \mathbb{R}^3$, let $\psi \in C^\infty(J \times \Omega)$ be a non-negative function such that $\int_{J\times\Omega} \psi \, dxdt = 1$ and also*

$$supp\psi \in \left\{(t,x)\colon |x|^2 < t, \quad 1 < t < 2\right\},$$

let $u \in L^2(J;V)$, let $\tilde{u}\colon \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be the function with the property that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ it holds that

$$\tilde{u}(t,x) = \begin{cases} u(t,x) & \text{if } \ (t,y) \in J \times \Omega \\ 0 \quad \text{otherwise}, \end{cases}$$

and let $\psi_\delta(u)$ be the function with the property that for all $(t,x) \in \mathbb{R} \times \mathbb{R}^3$ it holds that

$$\psi_\delta(u)(t,x) = \frac{1}{\delta^4} \int_{\mathbb{R}^4} \psi\Big(\frac{\tau}{\delta}, \frac{y}{\delta}\Big) \tilde{u}(t-\tau, x-y) \, dyd\tau.$$

Then the function $\psi_\delta(u)$ is called the retarded mollification of u.

Lemma 2.66 *Let $J \times \Omega$ be an open set in $\mathbb{R} \times \mathbb{R}^3$, let $u \in L^\infty(J;H) \cap L^2(J;V)$, let $\delta > 0$ and let $\psi_\delta(u)$ denote the retarded mollification of u. Then it holds that*

- $\operatorname{div} \psi_\delta(u) = 0$;
- *there exists a constant $C > 0$ such that*

$$\sup_{t \in J} \int_\Omega |\psi_\delta(u)|^2(t,x) \, dx \leq ess\ sup \int_\Omega |u|^2(t,x) \, dx;$$

- *and there exists a constant $C > 0$ such that*

$$\int_{J\times\Omega} |\nabla\psi_\delta(u)|^2(t,x)\,dxdt \leq C\int_{J\times\Omega} |\nabla u|^2(t,x)\,dxdt.$$

Proof (Main Idea) Let $N \in \mathbb{N}$, let $\delta = \frac{T}{N}$ and let $u_N, p_N\colon J\times\Omega \to \mathbb{R}^3$ be functions that solve the following PDE on $J\times\Omega$:

$$\partial_t U_N + \big(\psi_\delta(u_N)\cdot\nabla\big)u_N - \mu\Delta u_N + \nabla p + N = 0, \tag{2.41}$$

where ψ_δ is the retarded mollification operator 2.65. Note that by definition, $\psi_\delta(u_N)$ is smooth and its value at time t depends only on values of u prior to the time $t-\delta$.

Next, observe that for each $m \in \{0,1,\ldots,N-1\}$, Equation (2.41) is linear on the strip of the domain $\big(m\delta,(m+1)\delta\big)\times\Delta$. We can therefore obtain uniform in δ, a priori estimates for the functions u_N and p_N that are independent of N.

In particular, estimates on the pressure p_N, i.e., $p_N \in L^{5/3}(J\times\Omega)$ were obtained by H. Sohr and W. von Wahl in 1986 in [SvW86].

It can then be shown that the terms in the localised energy inequality are also bounded uniformly in δ. Finally, the Aubin-Lions Lemma A.17 can be used to obtain a strongly convergent subsequence that satisfies the properties $(i)-(iv)$. □

Theorem 2.61 shows that suitable weak solutions to the incompressible Navier-Stokes equations do indeed exist. We can therefore classify subsets associated with such solutions.

Definition 2.67 *Let $J\times\Omega$ be an open set in $\mathbb{R}\times\mathbb{R}^3$ and let $u\colon J\times\Omega\to\mathbb{R}^3$ be a suitable weak solution of the incompressible Navier-Stokes equations* (2.1) *in the domain $J\times\Omega$. The we denote by* $\mathrm{Reg}(u) \subseteq \mathbb{R}\times\mathbb{R}^3$ *the set given by*

$$\mathrm{Reg}(u) = \{(t,x)\in J\times\Omega\colon u\in L^\infty(V) \text{ in some neighbourhood of } (t,x)\},$$

and we call this set, the set of regular points of u.

Note that the set $\mathrm{Reg}(u)$ is relatively open.

Definition 2.68 *Let $J\times\Omega$ be an open set in $\mathbb{R}\times\mathbb{R}^3$, let $u\colon J\times\Omega\to\mathbb{R}^3$ be a suitable weak solution of the incompressible Navier-Stokes equations* (2.1) *in the domain* $J \times \Omega$ *and let* $\mathrm{Reg}(u)$ *denote the set of regular points of* u. *The we denote by* $\mathrm{Sing}(u) \subseteq \mathbb{R} \times \mathbb{R}^3$ *the set given by*

$$\mathrm{Sing}(u) = (J \times \Omega) \setminus \mathrm{Reg}(u),$$

and we call this set, the set of singular points of u.

In contrast to the set of regular points, the set $\mathrm{Sing}(u)$ is relatively closed.

Definition 2.69 *Let* $J \times \Omega$ *be an open set in* $\mathbb{R} \times \mathbb{R}^3$, *let* $u\colon J \times \Omega \to \mathbb{R}^3$ *be a suitable weak solution of the incompressible Navier-Stokes equations* (2.1) *in the domain* $J \times \Omega$ *and let* $\mathrm{Sing}(u)$ *denote the set of singular points of* u. *The we denote by* $\mathrm{Sing}_T(u) \subseteq \mathbb{R}$ *the set given by*

$$\mathrm{Sing}_T(u) = \left\{t\colon \{t\} \times \mathbb{R}^3 \cap \mathrm{Sing}(u) \neq \phi \right\},$$

and we call this set, the set of singular time points of u.

We can now state the main result of this section. Essentially, Caffarelli, Kohn and Nirenberg (see [CKN82]) were able to specify the Parabolic Hausdorff measure 2.56 of the set of singular points 2.68 of suitable weak solutions 2.60 to the incompressible Navier-Stokes equations (2.1) in three spatial dimensions.

Theorem 2.70 *Let* $J \times \Omega$ *be an open set in* $\mathbb{R} \times \mathbb{R}^3$ *and let* $u\colon J \times \Omega \to \mathbb{R}^3$ *be a suitable weak solution of the incompressible Navier-Stokes equations* (2.1) *in the domain* $J \times \Omega$. *Then the* 1*-Parabolic Hausdorff measure of the set of singular points of the solution* u *satisfies*

$$\mathcal{P}^1\big(\mathrm{Sing}(u)\big) = 0.$$

Remark 2.71 *Theorem 2.70 is an extension of a previous result due to V. Scheffer [Sch77]:*

Let $J \times \Omega$ *be an open set in* $\mathbb{R} \times \mathbb{R}^3$ *and let* $u\colon J \times \Omega \to \mathbb{R}^3$ *be a suitable weak solution of the incompressible Navier-Stokes equations* (2.1) *in the domain* $J \times \Omega$. *Then the* $5/3$*-Hausdorff measure of the set of singular points of the solution* u *satisfies*

$$\mathcal{H}^{5/3}\big(\mathrm{Sing}(u)\big) = 0,$$

and furthermore it holds that

$$\mathcal{H}^1\big(\mathrm{Sing}(u) \cap (\{t\} \times \Omega)\big) < \infty \qquad \text{uniformly in } t.$$

Corollary 2.72 *Let $J \times \Omega$ be an open set in $\mathbb{R} \times \mathbb{R}^3$ and let $u\colon J \times \Omega \to \mathbb{R}^3$ be a suitable weak solution of the incompressible Navier-Stokes equations* (2.1) *in the domain $J \times \Omega$.*

In view of Remark 2.57, a particular consequence of Theorem 2.70 is that

$$\mathcal{H}^{1/2}\big(\mathrm{Sing}_T(u)\big) = 0.$$

In fact, this result had already been been shown by Leray for the case $\Omega = \mathbb{R}^3$.

In order to prove Theorem 2.70, we require two additional propositions and an important lemma. For the sake of completeness, we present these results below.

Proposition 2.73 *Let the cylinders $Q_1 \subset \mathbb{R} \times \mathbb{R}^3$ and $Q_{1/2} \subset \mathbb{R} \times \mathbb{R}^3$ be defined as*

$$\begin{aligned} Q_1 &:= \{(\tau, y)\colon\ -1 < \tau < 1,\ |y| < 1\}\,, \\ Q_{1/2} &:= \{(\tau, y)\colon\ -1/4 < \tau < 1/4,\ |y| < 1/2\}\,, \end{aligned}$$

let $u\colon Q_1 \to \mathbb{R}^3$ be a suitable weak solution of the incompressible Navier-Stokes equations (2.1) *in the domain Q_1 with associated pressure field p and let $\epsilon > 0$ be a constant such that*

$$\int_{Q_1} \left(|u|^3 + |p|^{3/2}\right) dxdt \leq \epsilon_1,$$

Then there exists a constant $C_1 > 0$ such that for a.e. $(t, x) \in Q_{1/2}$ it holds that

$$|u(t, x)| \leq C_1.$$

Specifically, u is regular on the cylinder $Q_{1/2}$.

Proof Proposition 2.73 and its proof can be found in [CKN82]. □

Proposition 2.73 can then be used to prove the following proposition:

Proposition 2.74 *Let $r > 0$ be a constant, let the cylinder $Q_r \subset \mathbb{R} \times \mathbb{R}^3$ be defined as*

$$Q_r := \left\{(\tau, y)\colon\ -r^2 < \tau < r^2, |y| < r\right\},$$

let $u\colon Q_1 \to \mathbb{R}^3$ *be a suitable weak solution of the incompressible Navier-Stokes equations* (2.1) *in some neighbourhood of the point* $(t,x) \in Q_r$ *with associated pressure field* p *and let* $\epsilon_0 > 0$ *be a constant such that*

$$\limsup_{r\to 0} \frac{1}{r}\int_{Q_r} |\nabla u|^2 \, dxdt \leq \epsilon_0.$$

Then, the point $(t,x) \in \operatorname{Reg}(u)$*.*

Proof Proposition 2.74 and its proof can be found in [CKN82]. □

Finally, we require an analogue of the *Vitali covering lemma* for parabolic cylinders in $\mathbb{R}\times\mathbb{R}^3$.

Lemma 2.75 *Let* $\mathcal{I} \subseteq \mathbb{R}$ *be some uncountable index set and let* $g := \{Q_{r_i}(t_i,x_i)\}_{i\in\mathcal{I}} \subseteq \mathbb{R}\times\mathbb{R}^3$ *be a family of parabolic cylinders as defined in Definition 2.55 such that* g *is contained in a bounded subset of* $\mathbb{R}\times\mathbb{R}^3$*. Then there exists a countable subset* $\mathcal{J} \subset \mathcal{I}$ *such that the family of cylinders* $\tilde{g} \subseteq \mathbb{R}\times\mathbb{R}^3$ *given by*

$$\tilde{g} := \left\{Q_{r_j}(t_j,x_j)\colon j \in \mathcal{J}\right\},$$

satisfies the following:

(i) for all $i,j \in \mathcal{J}$ *with* $i \neq j$*, it holds that*

$$Q_{r_i}(t_i,x_i) \cap Q_{r_j}(t_j,x_j) = \phi;$$

(ii) and for all $Q \in g$ *there exists some* $Q_k(t_k,x_k) \in \tilde{g}$ *such that*

$$Q \subset Q_{5r_k}(t_k,x_k).$$

Proof The subfamily $\tilde{g}$ is constructed using induction. Let $g_0 = g$, let $\tilde{g}_0 = \phi$ and for each $n \in \mathbb{N}$ let Q_n denote the cylinder $Q_n := Q_{r_n}(t_n,x_n) \in g$. Then, for each $n \in \mathbb{N}$ we set

$$\tilde{g}_n = \tilde{g}_{n-1} \bigcup Q_n = \bigcup_{k=1}^{n} Q_n,$$

and

$$g_n = \{Q \in g\colon Q \cap Q_k = \phi \quad \text{for all } 1 \leq k \leq n\},$$

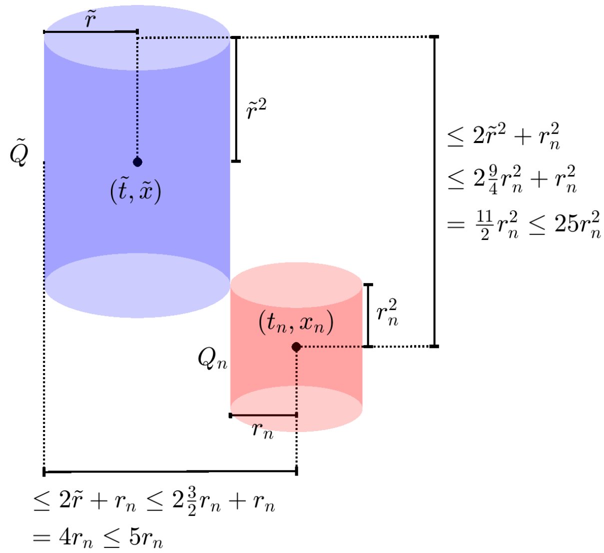


Figure 2.2: Geometric representation of the situation in the Vitali covering lemma 2.75.

where, for each $n \in \mathbb{N}$ the cylinder $Q_{n+1} := Q_{r_{n+1}}(t_{n+1}, x_{n+1}) \in g$ is selected such that

$$\forall Q := Q_r(t,x) \in g_n, \text{ it holds that } r \leq \frac{3}{2} r_{n+1}. \tag{2.42}$$

Thus, if for a given $n \in \mathbb{N}$ it holds that $g_n = \phi$, the process terminates and hence $\tilde{g} = \tilde{g}_n$ is finite. Otherwise, we have $\tilde{g} = \cup_{n=1}^{\infty} \tilde{g}_n$.

We now demonstrate that the family of subsets $\tilde{g}$ defined in this manner satisfies conditions (i) and (ii).

Clearly, (i) holds by definition of the families of subsets $\tilde{g}, \tilde{g}_n$. Furthermore, if $\tilde{g}$ is finite then (ii) also holds by definition.

Next, note that if $\tilde{g}$ is not finite, then by the boundedness assumption on the domain and Equation (2.42) it holds that $r_n \to 0$ as $n \to \infty$. Thus, given any $\tilde{Q} = R_r(t,x) \in g \setminus \tilde{g}$, there exists $n > 0$ such that $\tilde{Q} \in g_n$ and $\tilde{g} \not\subseteq g_{n+1}$. Hence, $\tilde{Q} \cap Q_{n+1} \neq \phi$ and therefore Equation (2.42) implies that $r \leq \frac{3}{2} r_{n+1}$ (see Figure 2.2).

Thus, $\tilde{Q} \subset Q_{5r_{n+1}}(t_{n+1}, x_{n+1})$, and therefore (ii) is also satisfied. □

We are now in a position to prove Theorem 2.70.

Proof (Theorem 2.70) Let $u\colon J \times \Omega \to \mathbb{R}^3$ be a suitable weak solution of the incompressible Navier-Stokes equations (2.1) in the domain $J \times \Omega$ with associated pressure field p.

Assume first that the domain $J \times \Omega$ is bounded and let the point $(t, x) \in \operatorname{Sing}(u)$. By Proposition 2.74 there exists a constant $\epsilon_0 > 0$ such that

$$\limsup_{r\to 0} \frac{1}{r} \int_{Q_r(t,x)} |\nabla u|^2 \, dxdt > \epsilon_0.$$

Next, let V be a neighbourhood of the the set $\operatorname{Sing}(u)$ and let $\delta > 0$. For each point $(t_i, x_i) \in \operatorname{Sing}(u)$, choose the cylinder $Q_{r_i}(t_i, x_i)$ with $r_i < \delta$ such that

$$\frac{1}{r_i} \int_{Q_{r_i}(t_i,x_i)} |\nabla u|^2 \, dxdt > \epsilon_0, \qquad \text{and } Q_{r_i}(t_i, x_i) \subset V. \tag{2.43}$$

Thus, we have obtained a family of parabolic cylinders, which we denote by g. By Lemma 2.75 there exists a countable subfamily of these cylinders $\tilde{g} := \{Q_j = Q_{r_j}(t_j, x_j),\ j \in \mathcal{J}\}$ with the property that for all $i, j \in \mathcal{J}$ with $i \neq j$, it holds that

$$Q_{r_i}(t_i, x_i) \cap Q_{r_j}(t_j, x_j) = \phi$$

and such that

$$\operatorname{Sing}(u) \subset \bigcup_{j\in\mathcal{J}} Q_{5r_j}(t_j, x_j). \tag{2.44}$$

Furthermore, by Inequality (2.43) it holds that

$$\begin{aligned} \sum_{j\in\mathcal{J}} r_j &\leq \frac{1}{\epsilon_0} \sum_{j\in\mathcal{J}} \int_{Q_j} |\nabla u|^2 \, dxdt \\ &\leq \frac{1}{\epsilon_0} \int_V |\nabla u|^2 \, dxdt, \end{aligned} \tag{2.45}$$

where the last inequality follows form the fact that the family of sets $\tilde{g}$ is pair-wise disjoint and contained in V.

Next, note that the Lebesgue measure $\mathcal{L}$ of the set $\cup_{i\in\mathcal{J}} Q_{5r_i}(t_i,x_i)$ satisfies

$$\mathcal{L}\Big(\bigcup_{i\in\mathcal{J}} Q_{5r_i}(t_i,x_i)\Big) \underbrace{\leq}_{\text{subadditivity}} \sum_{i\in\mathcal{J}} \mathcal{L}\big(Q_{5r_i}(t_i,x_i)\big) \leq \sum_{i\in\mathcal{J}} 2\pi(5r_i)^4$$

$$\underbrace{\leq}_{r_i<\delta} 2\pi 5^4\delta^3 \sum_{i\in\mathcal{J}} r_i \underbrace{\leq}_{\text{Inequality (2.45)}} 2\pi 5^4\delta^3 \frac{1}{\epsilon_0}\int_V |\nabla u|^2\,dx,$$

and the last term converges to 0 as $\delta\to 0$. It follows that

$$\mathcal{L}\Big(\bigcup_{i\in\mathcal{J}} Q_{5r_i}(t_i,x_i)\Big) \to 0 \text{ as } \delta\to 0,$$

and thus the inclusion (2.44) implies that the set $\operatorname{Sing}(u)$ has Lebesgue measure 0.

Next, note that Inequality (2.45) and the inclusion (2.44) together imply that for all neighbourhoods V of the set $\operatorname{Sing}(u)$ it holds that

$$\mathcal{P}^1\big(\operatorname{Sing}(u)\big) \leq \sum_{i\in\mathcal{J}} 5r_i \leq \frac{5}{\epsilon_0}\int_V |\nabla u|^2\,dxdt.$$

Finally, observe that the previously discussed energy estimates imply that the function ∇u is square-integrable. Together with the fact that $\mathcal{L}\big(\operatorname{Sing}(u)\big)$, this implies that we can choose neighbourhoods V with arbitrarily small Lebesgue measure. Thus, it holds that $\mathcal{P}^1\big(\operatorname{Sing}(u)\big) = 0$.

The general case for an unbounded domain follows from partitioning the domain into a countable number of bounded subdomains, showing that the result holds for each subdomain and then using the sub-additivity of the Hausdorff measure. The proof is thus complete. $\square$

2.4 Mild Solutions to the Incompressible Navier-Stokes Equations

Stokes System Revisited

Consider the setting of Definition 2.17 and let $\Omega = \mathbb{R}^d$. We recall that the *Stokes system* of equation is given by

$$\begin{aligned} u_t - \mu\Delta u + \nabla p &= f \qquad &&\text{in } (0,\infty)\times\Omega, \\ \operatorname{div} u &= 0 &&\text{in } (0,\infty)\times\Omega, \\ u(0,x) &= u_0(x). \end{aligned} \tag{2.46}$$

Note that without the imposition of additional constraints, the system of equations (2.46) does not have a unique solution. Indeed, given an arbitrary function $h\colon \mathbb{R}_+ \times \mathbb{R}^d$, we may set for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$

$$u(t,x) = \nabla h(t,x), \qquad \Delta h(t,x) = 0, \qquad p(t,x) = -h_t(t,x),$$

and obtain a solution to Equation (2.46). Note that if the forcing function f decays sufficiently fast as $x \to \infty$, then we may, for instance, impose the constraint that for all $t \in \mathbb{R}_+$ it holds that

$$u(t,x) \to 0 \text{ as } x \to \infty.$$

We now discuss some special cases of the Stokes system of equations (2.46):

Special Cases of Equation (2.46)

1. $f \equiv 0$:

   We set $p \equiv 0$. Then Equation (2.46) reduces to the well-known *Heat equation.* The solution to Equation (2.46) is thus explicitly given by

   $$u(t,x) = \int_{\mathbb{R}^d} \Gamma(t, x-y) u_0(y)\, dy = \Gamma(t) * u_0,$$

   where

   $$\Gamma(t,x) = \Big(\frac{1}{4\pi t}\Big)^{d/2} \exp\Big(\frac{-|x|^2}{4t}\Big).$$

2. $u_0 \equiv 0,\ f(t,x) = \nabla\phi(t,x)$:
   In this case, the solution to Equation (2.46) is given by

   $$u \equiv 0, \qquad p(t,x) = \phi(t,x) + c(t),$$

   where $c\colon \mathbb{R}_+ \to \mathbb{R}^d$ is some function of time.

3. $u_0 \equiv 0,\ \operatorname{div} f = 0$:
   The solution to Equation (2.46) is given by:

   $$\begin{aligned} u(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) f(y,s)\, dyds, \\ p(t,x) &= 0. \end{aligned}$$

4. $u_0 \equiv 0,\ \operatorname{div} f \neq 0$:
   The Helmholtz-Hodge decomposition 2.8 then implies that the there exists some scalar function $\phi\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ such that

   $$f = \mathbb{P}f + \nabla\phi,$$

   where $\mathbb{P}$ is the Leray Projector (c.f. Definition 2.9).
   The solution to Equation (2.46) is thus given by:

   $$\begin{aligned} u(t,x) &= \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) \mathbb{P}f(y,s)\, dyds, \\ \nabla p(t,x) &= \nabla\phi(t,x). \end{aligned}$$

Due to the linearity of the Stokes system of equations (2.46), the solution in the general case is given by a superposition of the solutions to the special cases $(1-4)$. Indeed, we have

$$\begin{aligned} u(t) &= \Gamma(t) * u_0 + \int_0^t \overbrace{\Gamma(t-s) * \mathbb{P}f(s)}^{K(t-s,x-y)f(y,s)}\, ds, \\ \nabla p &= f(t) - \mathbb{P}f(t). \end{aligned}$$

Here $*$ denotes the spatial convolution operator defined as follows: for functions $f, g\colon \mathbb{R}^d \to \mathbb{R}^d$, we have

$$f * g = \int_{\mathbb{R}^d} f(x-y)g(y)\,dy.$$

Next, we would like to re-write the function $f(t) \mapsto \int_0^t \Gamma(t-s) * \mathbb{P}f(s)\,ds$ in terms of a kernel K.

To this end, let G denote the fundamental solution of the Laplace operator and let $\Phi\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ be the function with the property that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ it holds that

$$\Phi(t,x) = \int_{\mathbb{R}^d} G(y)\Gamma(t, x-y)\,dy.$$

In particular, for $d \geq 3$, Φ is explicitly given by

$$\Phi(t,x) = \Big(\frac{1}{t}\Big)^{d-2/2} F\Big(\frac{|x|}{\sqrt{t}}\Big),$$

where $F\colon \mathbb{R}^d \to \mathbb{R}^d$ is a smooth function that decays $\sim r^{-(d-2)}$ as $r \to \infty$.

Furthermore, we recall that the Leray Projector 2.9 is the operator with the property that for all $x \in \mathbb{R}^d$ it holds that

$$\begin{aligned}\mathbb{P}(f)_{i,j}(x) &= f_i(x) - \frac{\partial}{\partial x_i}\big(\Delta^{-1} \operatorname{div} f\big) \\ &= f_i(x) + \sum_{j=1}^{d} \int_{\mathbb{R}^d} \frac{\partial^2}{\partial_{x_i}\partial_{x_j}} G(x-y) f_j(y)\,dy,\end{aligned}$$

where we have used the notation that $f := (f_1, f_2, \ldots, f_d)$. Thus, using the notation that $u = (u^1, u^2, \ldots, u^d), u_0 = (u_0^1, u_0^2, \ldots, u_0^d)$, we obtain that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ it holds that

$$\begin{aligned} u^i(t,x) = & \int_{\mathbb{R}^d} \Gamma(t, x-y) u_0^i(y)\, dy \\ & + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) \Big(f_i(x) + \sum_{j=1}^d \int_{\mathbb{R}^d} \frac{\partial^2}{\partial_{y_i} \partial_{y_j}} G(y-z) f_j(s,z) \Big)\, dz dy ds \\ = & \int_{\mathbb{R}^d} \Gamma(t, x-y) u_0^i(y)\, dy \\ & + \int_0^t \int_{\mathbb{R}^d} \underbrace{\int_{\mathbb{R}^d} \Gamma(t-s, x-y) G(y,z)}_{=\Phi(t-s, x-z)} \underbrace{\Big(-\Delta f_i(z) + \sum_{j=1}^d \frac{\partial^2}{\partial_{z_i} \partial_{z_j}} f_j(s,z) \Big)}_{\sum_{j=1}^d -\delta_{i,j} \Delta + \frac{\partial^2}{\partial_{z_i} \partial_{z_j}} f_j(s,z)}\, dz dy ds. \end{aligned}$$

Hence, we obtain that for each $i \in \{1, \ldots, d\}$ it holds that

$$\begin{aligned} u^i(t,x) = & \int_{\mathbb{R}^d} \Gamma(t, x-y) u_0^i(y)\, dy \\ & + \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^d \Big(-\delta_{i,j} \Delta + \frac{\partial^2}{\partial_{x_i} \partial_{x_j}} \Big) \Phi(t-s, x-y) f_j(s,y)\, dy ds \\ = & \int_{\mathbb{R}^d} \Gamma(t, x-y) u_0^i(y)\, dy + \int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^d K_{i,j}(t-s, x-y) f_j(s,y)\, dy ds. \end{aligned} \tag{2.47}$$

Here, we have introduced the kernel $\{K_{i,j}\}_{i,j=1}^d$ as the function with the property that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ it holds that

$$K_{i,j}(t,x) = -\delta_{i,j} \Delta + \frac{\partial^2}{\partial_{x_i} \partial_{x_j}} \Phi(t,x).$$

$\{K_{i,j}\}_{i,j=1}^d$ is known as the *Oseen kernel*. Before proceeding with our analysis, we state without proof some basic properties of the Oseen kernel.

Lemma 2.76 *Consider the setting of Definition 2.17 and let $K\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times d}$ denote the Oseen kernel. Then there exist constants $C, \tilde{C}$ such* *that for all $l, k \in \mathbb{N}$ and for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ it holds that*

$$|K_{i,j}(t,x)| \leq C \frac{1}{\big(|x|^2+t\big)^{d/2}},$$
$$|\nabla_t^l \nabla_x^k k_{i,j}(t,x)| \leq \tilde{C} \frac{1}{\big(|x|^2+t\big)^{\frac{d+k+2l}{2}}}.$$

Returning to Equation 2.47, observe that if the forcing function f can be written as $f = \operatorname{div} F$, for some function $F = \{F_{i,j}\}_{i,j=1}^d$, then we may re-write Equation (2.47) in the compact form

$$u^i(t,x) = \int_{\mathbb{R}^d} \Gamma(t, x-y) u_0^i(y)\, dy + \int_0^t \int_{\mathbb{R}^d} \sum_{j,l=1}^d \bar{K}_{i,j,l}(t-s, x-y) F_{j,l}(s,y)\, dyds,$$

where

$$\bar{K} := \{\bar{K}_{i,j,l}\}_{i,j,l=1}^d = \nabla K.$$

It can then be shown, using the bounds obtained from Lemma 2.76, that for any $u_0 \in L^\infty(\mathbb{R};\mathbb{R}^d)$, $F \in L^\infty\big([0,T]\times\mathbb{R}^d;\mathbb{R}^d\big)$ and any $\theta \in (0,1), R > 0, \delta > 0$ it holds that

$$u \in C^\theta\big([\delta,T]\times B_R;\mathbb{R}^d\big),$$

and there exists some constant C such that

$$\|u\|_{C^\theta\left(B_R\times[\delta,T]\right)} \leq C(R, \delta, \|u_0\|, \|F\|_{L^\infty}).$$

Let us now consider the following initial value problem involving the normalised incompressible Navier-Stokes equations with kinematic viscosity $\mu \equiv 1$:

$$\begin{aligned} u_t + \nabla p - \Delta u &= -\operatorname{div}(u\otimes u), && \text{in } (0,\infty)\times\mathbb{R}^d,\\ \operatorname{div} u &= 0, && \text{in } (0,\infty)\times\mathbb{R}^d,\\ u(x,0) &= u_0(x). \end{aligned} \tag{2.48}$$

We are now ready to define *mild* solutions to the the IVP (2.48).

Definition 2.77 *Consider the IVP* (2.48)*, let* $T \in (0, \infty), d \in \mathbb{N}$*, let* $u_0 \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$ *and let* $u\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ *be a function with the property that for all* $t \in [0,T]$ *it holds that*

$$u(t, \cdot) = \Gamma(t) * u_0 + \int_0^t \bar{K}(t-s) * \big(- u(s) \otimes u(s)\big)\, ds, \tag{2.49}$$

where

$$\Gamma(t,x) = \Big(\frac{1}{4\pi t}\Big)^{d/2} \exp\Big(\frac{-|x|^2}{4t}\Big), \tag{2.50}$$

$$\bar{K}(t,x) := \{\bar{K}_{i,j,l}(t,x)\}_{i,j,l=1}^d = \nabla K(t,x), \tag{2.51}$$

$$K(t,x) := \{K_{i,j}(t,x)\}_{i,j=1}^d = -\delta_{i,j}\Delta + \frac{\partial^2}{\partial_{x_i}\partial_{x_j}}\Phi(t,x). \tag{2.52}$$

The existence of mild solutions to the IVP involving the incompressible Navier-Stokes equations (2.48) can be demonstrated. Let us begin by introducing some notation.

For each $t \in [0,T]$, we denote by $U(t)$ the spatial convolution $\Gamma(t) * u_0$ and for each function $u, v\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and each time $t \in [0,T]$ we denote by $B(u,v)(t)$ the function given by

$$B(u,v)(t) = \int_0^t K(t-s) * \big(- u(s) \otimes v(s)\big)\, ds.$$

Using this notation, we may write Equation (2.49) in the more revealing form

$$u = U + B(u,u). \tag{2.53}$$

Equation (2.53) closely resembles a fixed point equation. We may therefore appeal to the following fixed point theorem to obtain the existence of a unique solution:

Lemma 2.78 *Let* X *be a Banach space, let* $B\colon X \times X \to X$ *be a continuous, bilinear form on* X *with the property that there exists some constant* $\gamma > 0$ *such that for all* $x, y \in X$ *it holds that*

$$\|B(x,y)\| \leq \gamma \|x\|_X \|y\|_X,$$

and let $a \in X$ *such that* $4\gamma\|a\|_X < 1$*. Then the equation*

$$x = a + B(x,x),$$

has a unique solution $\bar{x}$ in the ball

$$\left\{x \in X \colon |X|_X < \frac{1+\sqrt{1-4\gamma\|a\|_X}}{2\gamma}\right\},$$

and moreover $\bar{x}$ satisfies the bound

$$\|\bar{x}\|_X \leq \frac{1+\sqrt{1-4\gamma\|a\|_X}}{2\gamma}.$$

Proof The proof, which follows from the use of Picard iterations, is left as an exercise. □

We can now apply Lemma 2.78 for the Banach space $X = X_T = L^\infty\big((0,T)\times \mathbb{R}^d;\mathbb{R}^d\big)$, which allows us to state an existence and uniqueness result for mild solutions to the IVP involving the incompressible Navier-Stokes equation (2.48).

Theorem 2.79 (Leray) *Let $u_0 \in L^\infty(\mathbb{R}^d;\mathbb{R}^d)$. Then there exists some time $T_0 > 0$ depending on $\|u\|_{L^\infty}$ and a unique solution $u \in C^\infty\big((0,T_0] \times \mathbb{R}^d;\mathbb{R}^d\big) \cap L^\infty\big((0,T_0] \times \mathbb{R}^d;\mathbb{R}^d\big)$ to the IVP* (2.48).

Proof (Sketch) Consider the setting of Lemma 2.78 and let the space $X = X_T = L^\infty\big((0,T) \times \mathbb{R}^d;\mathbb{R}^d\big)$. The maximum principle for the heat equation implies that

$$\|U\| \leq \|u_0\|_{L^\infty}.$$

Furthermore, for any $u, v \in X_T$, it holds that

$$\|B(u,v)\|_{X_T} \leq \|u\|_{X_T}\|v\|_{X_T} \int_0^T \int_{\mathbb{R}^d} |\bar{K}(t,x)|\, dxdt.$$

Note that lemma 2.76 implies that there exists some constant $C > 0$ such that for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$ it holds that

$$|\nabla K_{i,j}(t,x)| \leq C \frac{1}{\big(|x|^2+t\big)^{\frac{d+1}{2}}},$$

and therefore it holds that

$$\int_0^T \int_{\mathbb{R}^d} |\bar{K}(t,x)|\, dxdt \leq C \int_0^T \int_{\mathbb{R}^d} \frac{1}{\big(|x|^2+t\big)^{\frac{d+1}{2}}}\, dxdt \leq \tilde{C}\sqrt{T},$$

where $\tilde{C} > 0$ and the last inequality has been left as an exercise. It therefore follows that there exists some constant $\tilde{C} > 0$ such that for all $u, v \in X_T$ it holds that

$$\|B(u,v)\|_{X_T} \leq \tilde{C}\sqrt{T}\|u\|_{X_T}\|v\|_{X_T}$$

Thus, if we pick $T^* = T$ sufficiently small such that $\tilde{C}\sqrt{T^*}\|u\|_{L^\infty} \leq \frac{1}{4}$, we can apply Lemma 2.78 to obtain the existence of a mild solution $u \in X_T$ up to time T^*. The uniqueness of the mild solution can be obtained by together individual solutions and the regularity can be proven using Hölder continuity estimates. □

Remark 2.80 *Several remarks are now in order.*

- *In general, it does not hold that*

$$\|u(t) - u_0\|_{L^\infty}.$$

- *If $T^* > 0$ is the maximal time of existence, then it can be shown that there exists some constant $\epsilon_1 > 0$ such that*

$$\|u(t)\|_{L^\infty} \geq \frac{\epsilon_1}{\sqrt{T^* - t}} \quad \text{as } t \to T^*_-$$

- *Any mild solution $u \in X_T$ is smooth in $(0, T^*) \times \mathbb{R}^d$ with $t^{k/2}\nabla^k u(t) \in X_T$.*

- *The above construction of the mild solution $u \in X_T$ is completely independent of the energy identity.*

In general, other choices of the space X_T are also possible, although the existence and uniqueness proof for solutions $u \in X_T$ becomes more involved. The interested reader can, for example, consult [FK64] and [Kat84].

2.4.1 Blow-up Criteria for Mild Solutions

We briefly discuss (without stating a proof) Serrin's blow-up criterion [Ser62].

Theorem 2.81 (Serrin) *Consider the IVP (2.48), let $T, T^* \in (0, \infty), d \in \mathbb{N}$, let $u\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be a mild solution to the IVP (2.48) such that u blows up at time T^*, let $q > d$ and let $p \geq 2$ be such that*

$$\frac{2}{p} + \frac{d}{q} = 1.$$

Then for all $\tau > 0$ it holds that

$$\int_{T-\tau}^{T} \Big(\int_{\mathbb{R}^d} |u(t,x)|^q \, dx \Big)^{p/q} dt = +\infty$$

Remark 2.82 *In the case of two spatial dimensions, i.e., $d = 2$, if we assume that the solution $u \in X_T$ satisfies the energy identity*

$$\int_0^T \int_{\mathbb{R}^d} |u(t,x)|^4 \, dxdt \leq C < \infty,$$

for some constant C, then we can pick $q = 4, p = 4$ and obtain that the solution u does not blow-up and $T^* = \infty$.

Remark 2.83 *As shown recently by Escauriaza, Seregin and Sverak [ESS03] the choice of $d = 3, q = d$ works as well.*

Remark 2.84 *A related result is Serrin's interior regularity theorem [Ser62]. Let $u \in L^{\infty;2}_{t,x} \cap L^2_t H^1_x$ be a weak solution of the incompressible Navier-Stokes equations and suppose in addition that $p, q \in \mathbb{N}$ are such that $\frac{2}{p} + \frac{d}{q} < 1$ and also u*
in$L^{p,q}_{t,x}$. Then for each $t \in [0,T]$, it holds that $u \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and furthermore, if u is strongly differentiable with respect to t, then both u and $\nabla^k_x u$ are absolutely continuous with respect to time.

We conclude this section by stating a further improvement of the Serrin blow-up criterion 2.81.

Theorem 2.85 (Struwe) *Consider the IVP (2.48), let $T, \in (0,\infty), d \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^d$ be an open domain and let $u \colon [0,T] \times \Omega \to \mathbb{R}^d$ be a mild solution to the IVP (2.48) on the domain Ω such that one of the following conditions holds:*

1. *there exist constants $p, q \in \mathbb{N}$ such that $q > d$, $\frac{2}{p} + \frac{d}{q} \leq 1$ and such that $u \in L^p\big(0,T; L^q(\mathbb{R}^d)\big)$;*
2. *or $u \in L^\infty\big(0,T; L^d(\mathbb{R}^d)\big)$ and there exists some absolute constant $\epsilon > 0$ such that for all $t \in [0,T]$, there exists $R > 0$ with the property that*

$$\int_{B_R(x) \cap \Omega} |u(t,x)|^d \, dx \leq \epsilon.$$

Then the solution $u \in L^\infty\big([0,T] \times \Omega\big)$.

Proof (Heuristics) Theorem 2.85 and its proof can be found in [Str88, Theorem 3.1]. We instead present a heuristic argument.

Let us consider a mild solution u to the IVP (2.48) with initial condition $u_0 \in L^\infty(\mathbb{R}^d; \mathbb{R}^d) \cap L^2(\mathbb{R}^d; \mathbb{R}^d)$.

Let $(0, T^*)$ be the maximal interval of existence of the mild solution and assume that $0 < T^* < \infty$. Then, Remark 2.80 implies that $\|u(t)\|_{L^\infty} \to \infty$ as $t \to T^*_-$.

Let us therefore consider a sequence of real numbers $0 < M_1 < M_2 < \ldots \to \infty$ with M_1 sufficiently large. For $j \in \mathbb{N}$ we denote by t_j the first time such that $\|u(t_j)\|_{L^\infty}$ takes the value M_j. Next, let $x_j \in \mathbb{R}^d$ be such that $|u(t_j, x_j)| = M_j$. It follows by the definition of the sequence $\{t_j\}_{j\in\mathbb{N}}$ that for all $x \in \mathbb{R}^d$, for all $j \in \mathbb{N}$ and for all $t \in [0, t_j]$, it holds that

$$|u(t, x)| \leq M_j.$$

In addition, it can be shown that the sequence $\{x_j\}_{j\in\mathbb{N}}$. Next, we define for each $j \in \mathbb{N}$ the function $v_j \colon [-M_j^2 t_j, M_j^2(T^* - t_j)] \times \mathbb{R}^d \to \mathbb{R}^d$ given by

$$v_j(s, y) = \frac{1}{M_j} u\Big(t_j + \frac{s}{M_j^2}, x_j + \frac{y}{M_j}\Big).$$

Clearly, for all $s, y \in [-M_j^2 t_j, 0] \times \mathbb{R}^d$ it holds that

$$|v_j(s, y)| \leq 1,$$

and in particular it holds that

$$|v_j(0, 0)| = 1.$$

Next, let $\rho > 0$ be a fixed constant. It is possible to use Hölder estimates, which we do not prove, to conclude that for all $(t, x) \in [-\rho^2, \rho^2] \times B_\rho$ it holds that

$$|v_j(t, x)| \geq \frac{1}{2}.$$ □

For clarity of exposition, for each $j \in \mathbb{N}$, let Z_j denote (t_j, x_j) and let Q_j denote $Q_{z_j, \frac{\rho}{M_j}} := (t_j - \frac{\rho^2}{M_j^2}, t_j) \times B_{x_j, \frac{\rho}{M_j}}$. Then, we may use rescaling to conclude that for all $(t, x) \in Q_j$ it holds that

$$|u(t, x)| \geq \frac{M_j}{2},$$

and therefore there exist constants $C, \tilde{C} < 0$ such that

$$\int_{Q_j} |u(t,x)|^{d+2}\, dxdt \geq |\frac{M_j}{2}|^{d+2}|Q_j| \geq CM_j^{d+2}\frac{\rho^2}{M_j^2}|\frac{\rho}{M_j}|^d \geq \tilde{C}\rho^{d+2}.$$

Next, we can assume without loss of generality (by passing to a subsequence if necessary) that the sequence $\{Q_j\}_{j\in\mathbb{N}}$ is disjoint. It therefore follows that for all $\tau > 0$ it holds that

$$\int_{T-\tau}^{T}\int_{\mathbb{R}^d} |u|^{d+2}\, dxdt \geq \bigcup_{j\in\mathbb{N}}\int_{Q_j} |u|^{d+2}\, dxdt = \infty.$$

Moreover, for $q = d+2 = p$, we obtain for each $j \in \mathbb{N}$

$$\left(\int_{t_j-\frac{\rho^2}{M_j^2}}^{t_j} \Big(\int_{B_{\frac{\rho}{M_j}}(x_j)} |u(t,x)|^q\, dx\Big)^{p/q} dt\right)^{1/p}$$

$$\geq \left(\int_{t_j-\frac{\rho^2}{M_j^2}}^{t_j} \Big(\int_{B_{\frac{\rho}{M_j}}(x_j)} |\frac{M_j}{2}|^q\, dx\Big)^{p/q} dt\right)^{1/p}$$

$$= \frac{M_j}{2}\Big(\int_{t_j-\frac{\rho^2}{M_j^2}}^{t_j} \big(\frac{\rho}{M_j}\big)^{d\frac{p}{q}}\, dt\Big)^{1/p}$$

$$= \frac{M_j}{2}\big(\frac{\rho}{M_j}\big)^{d/q}\big(\frac{\rho^2}{M_j^2}\big)^{1/p}$$

$$= \frac{1}{2}\rho^{\frac{d}{q}+\frac{2}{p}}M_j^{1-\frac{d}{p}-\frac{2}{p}} = \frac{1}{2}\rho.$$

It therefore follows that

$$\|u\|_{L^{p,q}_{t,x}(Q_T)} \geq \bigcup_{j\in\mathbb{N}} \|\|_{L^{p,q}_{t,x}(Q_j)} = \infty.$$

The proof is thus complete.

3 The Incompressible Euler Equations

We recall from Chapter 2 that the incompressible Navier-Stokes equations for a Newtonian fluid (**INS**) in non-dimensionalised form are given by

$$u_t + (u \cdot \nabla)u + \nabla p = \frac{1}{\mathrm{Re}}\Delta u \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{3.1a}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega \times \mathbb{R}_+. \tag{3.1b}$$

Here, $d \in \{2,3\}$ and $\Omega \in \mathbb{R}^d$ is the domain of interest, $u \in \mathbb{R}^d$ is the velocity field and $\nabla p \in \mathbb{R}^d$ is the gradient of the scalar pressure field:

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix}, \qquad \nabla p = \begin{bmatrix} \partial_{x_1} p \\ \partial_{x_2} p \\ \vdots \\ \partial_{x_d} p \end{bmatrix},$$

$\operatorname{div} u$ and Δu are the divergence and Laplacian respectively of the velocity field u and are given by

$$\operatorname{div} u = \sum_{j=1}^{d} \frac{\partial u}{\partial x_j}, \qquad \Delta u = \sum_{j=1}^{d} \frac{\partial^2 u}{\partial x_j^2},$$

the vector operator $(u \cdot \nabla)u \in \mathbb{R}^d$ is given by

$$(u \cdot \nabla)u = \begin{bmatrix} \sum_{i=1}^{d} u_i \partial_{x_i} u_1 \\ \sum_{i=1}^{d} u_i \partial_{x_i} u_2 \\ \vdots \\ \sum_{i=1}^{d} u_i \partial_{x_i} u_d \end{bmatrix},$$

and we have used the notation $u_t := \frac{\partial u}{\partial t}$, $\partial_{x_i} p := \frac{\partial p}{\partial x_i}$ and $\partial_{x_i} u_j := \frac{\partial u_j}{\partial x_i}$.

Note that the only parameter in Equation (3.1) is the Reynolds number

$$\mathrm{Re} = \frac{UL}{\nu},$$

where U and L are reference velocity and length scales respectively and ν is the kinematic viscosity.

For several flows of interest, the Reynolds number Re is typically very large, ranging from $\mathrm{Re} = 10^5 - 10^6$ for hydrological flows to $\mathrm{Re} = 10^{15} - 10^{20}$ for astrophysical flows. It is thus of physical interest to consider flows in the zero viscosity limit, i.e., in the limit $\mathrm{Re} \to \infty$.

Let us denote by u^{Re}, solutions of Equation (3.2). Then under the assumption that these solutions are uniformly smooth, it can be shown that

$$u^{\mathrm{Re}} \to u \quad \text{as } \mathrm{Re} \to \infty, \quad \text{in the appropriate topology},$$

where $u \in \mathbb{R}^d$ is the solution to the incompressible Euler equations (**ICE**) given by

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= 0, \\ \operatorname{div} u &= 0. \end{aligned} \tag{3.2}$$

Note however, that this convergence result, which is based on energy methods, holds only if the domain $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$ (i.e., periodic boundary conditions). Indeed, the result is not necessarily true in the case of non-trivial bounded domains Ω where the boundary layers play an important role. Throughout the remainder of this chapter therefore, we assume that $\Omega = \mathbb{R}^d$, i.e., the entire space or $\Omega = \mathbb{T}^d$, i.e, a domain with periodic boundary conditions.

Equation (3.2) is the so-called *velocity-pressure* formulation of the incompressible Euler equations. Note that the pressure term in Equation (3.2) can be eliminated using the so-called *Pressure-Poisson* equation (cf., Equation (2.5)):

$$-\Delta p = \operatorname{div}\big((u \cdot \nabla)u\big). \tag{3.3}$$

In the remainder of this chapter, we study alternative formulations of the incompressible Euler equations (3.2).

3.1 Vorticity-Stream function formulation of the incompressible Euler equations

Throughout this section, unless stated otherwise, we assume that all associated functions including the velocity field are smooth. We now begin by introducing the vorticity of a velocity field.

Definition 3.1 (Vorticity) *Let $T \in (0, \infty]$, $d \in \{2, 3\}$, let $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$, and let $u \colon \Omega \times [0, T) \to \mathbb{R}^d$ be a velocity vector field. Then the vorticity ω of the velocity u is defined as $\omega = \operatorname{curl} u$.*

We can now explicitly differentiate Equation (3.2) to obtain

$$\omega_t + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u, \tag{3.4}$$

or equivalently,

$$\frac{\mathrm{D}\omega}{\mathrm{Dt}} = (\omega \cdot \nabla)u,$$

where $\frac{\mathrm{D}}{\mathrm{Dt}}$ is the material derivative introduced in Chapter 1.

Of course, depending on whether the spatial dimension $d = 2$ or $d = 3$, we obtain two distinct formulations of Equation (3.4).

3.1.1 Vorticity-Stream function formulation in 2-D

Consider the setting of Definition 3.1, let $d = 2$ and let $\Omega = \mathbb{R}^2$. This implies that our domain is the entire space $\mathbb{R}^2$, and the velocity field u is restricted to a planar flow:

$$u = (u_1, u_2, 0),$$

so that

$$\omega = \operatorname{curl} u = (0, 0, \partial_{x_1} u_2 - \partial_{x_2} u_1),$$

and thus, the vorticity ω is a *scalar* function.

Furthermore, since $\omega \perp u$, it is straightforward to show that

$$(\omega \cdot \nabla)u \equiv 0 \quad \text{in 2-D.}$$

Therefore, the vorticity formulation of the incompressible Euler equations (3.2) in 2-D is a *scalar* equation given by

$$\omega_t + (u \cdot \nabla)\omega = 0, \tag{3.5}$$

or equivalently,

$$\frac{\mathrm{D}\omega}{\mathrm{Dt}} = 0.$$

Next, we introduce streamlines i.e., Lagrangian trajectories (see Section 1.2) as functions $\mathcal{X} \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ that satisfy the following initial value problem:

$$\begin{aligned} \frac{d\mathcal{X}}{dt}(a,t) &= u\big(\mathcal{X}(a,t),t\big), \\ \mathcal{X}(a,0) &= a, \end{aligned} \tag{3.6}$$

where $a \in \Omega$.

Clearly, $\frac{\mathrm{D}\omega}{\mathrm{dt}} \equiv 0$ implies that for all $a \in \Omega$ it holds that

$$\omega\big(\mathcal{X}(a,t),t\big) = \omega\big(\mathcal{X}(a,0),0\big) := \omega_0(a), \tag{3.7}$$

where $\omega_0 \colon \Omega \to \mathbb{R}$ is the initial vorticity. In other words, the vorticity remains constant along particle trajectories in 2-D. This fact considerably simplifies the analysis of flows in two dimensions. Note that 3-D flows are considerably more complicated and, as we will discuss in the next section, the vorticity may not be conserved along particle trajectories.

Unfortunately, Equation (3.5) still involves the unknown velocity field u. Therefore, in order to solve Equation (3.5), we must eliminate the velocity term and obtain an evolution equation purely in terms of the vorticity ω.

To this end, we recall that by assumption $\operatorname{div} u \equiv 0$. Hence, by the Helmholtz-Hodge decomposition theorem 2.8 there exists a (up to an additive constant) unique divergence-free scalar field $\psi \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ with the property that

$$u_1 = -\partial_{x_2}\psi, \qquad u_2 = \partial_{x_1}\psi.$$

Indeed, an explicit computation shows that $\operatorname{div} \psi \equiv 0$.

The function ψ is now known as the *stream function*. Next, using the definition of the vorticity in 2-D, we obtain that

$$\omega = \partial_{x_1} u_2 - \partial_{x_2} u_1 = \partial^2_{x_1 x_1}\psi + \partial^2_{x_2 x_2}\psi = \Delta\psi.$$

Hence, the vorticity ω is given by the following Poisson equation:

$$\Delta\psi = \omega \qquad \text{in } \Omega \tag{3.8}$$

Under the assumption that all associated functions decay sufficiently fast as $|x| \to \infty$, the Poisson equation (3.8) can be explicitly solved in terms of the Green's function:

$$\psi(x,t) = \frac{1}{2\pi}\int_\Omega \log\big(|x-y|\big)\omega(y,t)\,dy. \tag{3.9}$$

Moreover, given the definition of $u = (u_1, u_2)$ and the smoothness assumption, Equation (3.9) can be explicitly differentiated to obtain

$$u(x,t) = \int_\Omega K_2(x-y)\omega(y,t)\,dy, \tag{3.10}$$

where the kernel $K_2\colon \Omega \to \mathbb{R}^2$ is the function with the property that for all $x = (x_1, x_2) \in \Omega$ it holds that

$$K_2(x) = \frac{1}{2\pi}\big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\big).$$

Therefore, in view of the preceding calculations, we arrive at the following vorticity-stream function formulation for the 2-D incompressible Euler equation (3.2):

Lemma 3.2 *Let $T \in (0,\infty]$, let $d = 2$, let $\Omega = \mathbb{R}^d$ and let $u\colon \Omega \times [0,T) \to \mathbb{R}^d$ be a velocity vector field with the property that u vanishes rapidly as $|x| \to \infty$. Then the velocity-pressure formulation of the 2-D incompressible Euler equations* (3.2) *is equivalent to the following vorticity-stream function formulation:*

$$\frac{D\omega}{dt} = \partial_t\omega + (u\cdot\nabla)\omega \equiv 0 \qquad \text{for all } (x,t) \in \mathbb{R}^d \times (0,T),$$
$$\omega(x,0) = \omega_0(x) \quad \text{for all } x \in \mathbb{R}^d,$$

where the velocity u is given by

$$u(x,t) = \int_\Omega K_2(x-y)\omega(y,t)\,dy,$$

with kernel K_2 given by

$$K_2(x) = \frac{1}{2\pi}\big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\big).$$

Moreover, the pressure field p can be obtained by solving the Poisson equation (3.3)*.*

3.1.2 Periodic Boundary Conditions in 2-D

The analysis in Section 3.1.1 was limited to the case of $\Omega = \mathbb{R}^2$. Let us now assume that the domain $\Omega = \mathbb{T}^d$, i.e., we assume a domain with periodic boundary conditions. Most of the analysis in Section 3.1.1 is still valid and in particular, it holds that the vorticity ω satisfies the Poisson equation (3.8) given by

$$\Delta\psi = \omega \qquad \text{in } \Omega.$$

Since $\Omega = \mathbb{T}^d$, we can readily solve Equation (3.8) using Fourier series expansion. Indeed, let the vorticity ω be given by

$$\omega(x,t) = \sum_k \hat{\omega}_k(t) e^{2\pi i k\cdot x}.$$

Then the Poisson equation (3.8) has an explicit, periodic solution given by

$$\psi(x,t) = -\sum_{k\neq 0} \frac{1}{4\pi|k|^2} \hat{\omega}_k(t) e^{2\pi i x\cdot k},$$

provided that the additional constraint

$$\int_\Omega \omega \, dx \equiv 0,$$

is satisfied. Note that since by definition, the vorticity $\omega = \partial_{x_1} u_2 - \partial_{x_2}, u_1$ and is periodic, this additional constraint is indeed satisfied.

Unfortunately however, the velocity field $u = (u_1, u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$, and it therefore follows that

$$\int_\Omega u \, dx = 0.$$

In other words, the velocity field must necessarily have zero-mean. In order to consider flows with non-zero mean, we can use the mean-value decomposition:

$$u = \bar{u} + \tilde{u},$$

where

$$\bar{u} = \int_\Omega u \, dx, \qquad \text{is a constant},$$

and

$$\int_\Omega \tilde{u}\,dx \equiv 0.$$

We can now compute the zero-mean velocity field $\tilde{u}$ from the stream function ψ by setting for all $(x,t) \in \Omega \times [0,T)$

$$\tilde{u}(x,t) = \sum_{k\neq 0} \frac{(-k_2,k_1)}{2\pi i|k|^2}\hat{\omega}(k,t)e^{2\pi i x\cdot k}.$$

We therefore obtain the following lemma regarding the vorticity-stream function formulation for the 2-D incompressible Euler equation with periodic boundary conditions:

Lemma 3.3 *Let $T \in (0,\infty]$, let $d = 2$, let $\Omega = \mathbb{T}^d$, let $u_0\colon \Omega \to \mathbb{R}^d$ be an initial velocity field and let $u\colon \Omega \times [0,T) \to \mathbb{R}^d$ be a velocity vector field with mean-value decomposition $u = \bar{u}_0 + \tilde{u}$ where $\bar{u}_0 = \int_\Omega u_0\,dx, \operatorname{div} u \equiv 0$. The the velocity-pressure formulation of the 2-D incompressible Euler equations* (3.2) *with periodic boundary conditions is equivalent to the following vorticity-stream function formulation:*

$$\begin{aligned} \partial_t\omega + \big((\bar{u}_0 + \tilde{u})\cdot\nabla\big)\omega \equiv 0 \qquad & \text{for all } (x,t) \in \mathbb{T}^d \times (0,T), \\ \omega(x,0) = \omega_0(x) \qquad & \text{for all } x \in \mathbb{R}^d, \end{aligned}$$

where the velocity field $\tilde{u}$ is given by

$$\tilde{u}(x,t) = \sum_{k\neq 0} \frac{(-k_2,k_1)}{2\pi i|k|^2}\hat{\omega}(k,t)e^{2\pi i x\cdot k}. \tag{3.11}$$

Once again, the pressure field p can be obtained by solving the Poisson equation (3.3)*.*

3.1.3 Vorticity-Stream function formulation in 3-D

Consider the setting of Definition 3.1, let $d = 3$ and let $\Omega = \mathbb{R}^3$. This implies that our domain is the entire space $\mathbb{R}^3$ and the velocity field u is given by

$$u = (u_1, u_2, u_3),$$

so that

$$\omega = \operatorname{curl} u = (\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_3} u_1 - \partial_{x_1} u_3, \partial_{x_1} u_2 - \partial_{x_2} u_1),$$

and thus, the vorticity ω is now a *vector* function.

The vorticity formulation of the incompressible Euler equations (3.2) in 3-D can then be derived as

$$\begin{aligned} \omega_t + (u \cdot \nabla)\omega &= (\omega \cdot \nabla) u, \\ \omega(x, 0) &= \omega_0(x). \end{aligned} \tag{3.12}$$

Note that in contrast to the case of two spatial dimensions, the fact that $\omega \in \mathbb{R}^3$ implies that Equation (3.12) is a system of three equations. Unfortunately, similar to the case of two spatial dimensions, Equation (3.12) also involves the unknown velocity field u. Therefore, in order to solve Equation (3.5), we must once again eliminate the velocity term using a suitable stream function and thus obtain evolution equations purely in terms of the vorticity ω.

To this end, we recall that by assumption we have

$$\begin{aligned} \operatorname{div} u &\equiv 0, \\ \operatorname{curl} u &= \omega. \end{aligned} \tag{3.13}$$

The system of Equations (3.12)-(3.13) is overdetermined, and we can once again appeal to the Helmholtz-Hodge decomposition theorem 2.8. Indeed, we have the following result [MB02, Proposition 2.16]:

Theorem 3.4 *Let* $T \in (0, \infty]$, *let* $d = 3$, *let* $\Omega = \mathbb{R}^d$, *let* $u\colon \Omega \times [0, T) \to \mathbb{R}^d$ *be a velocity vector field and let* $\omega\colon [0, T) \to L^2(\Omega; \mathbb{R}^d)$ *be a smooth vector field. Then*

(i) Equation (3.13) *has a unique, smooth solution* u *that vanishes rapidly as* $|x| \to \infty$ *if and only if*

$$\operatorname{div} \omega \equiv 0;$$

(ii) if $\operatorname{div} \omega \equiv 0$, *then the solution* u *can be determined constructively as*

$$u = -\operatorname{curl} \psi,$$

where the vector-stream function ψ *solves the following Poisson equation:*

$$\Delta \psi = \Omega.$$

Moreover, the velocity field u is explicitly given by

$$u(x,t) = \int_\Omega K_3(x-y)\omega(y,t)\,dy$$

with the 3×3 matrix kernel K_3 given by

$$K_3(x)h = \frac{1}{4\pi}\frac{x \times h}{|x|^3}, \qquad \forall\ h \in \mathbb{R}^3.$$

Proof We first recall two vector identities from multi-variable calculus: let $\psi \in \mathbb{R}^3$. Then it holds that

$$\begin{aligned} \operatorname{div}\operatorname{curl}\psi &\equiv 0, \\ -\operatorname{curl}\operatorname{curl}\psi + \nabla\operatorname{div}\psi &= \Delta\psi. \end{aligned} \tag{3.14}$$

Let u be the unique, smooth solution to Equation (3.13). Then, clearly

$$\operatorname{div}\omega = \operatorname{div}\operatorname{curl}u \equiv 0.$$

Conversely, let $\operatorname{div}\Omega \equiv 0$. Then, by the Helmholtz-Hodge decomposition theorem 2.8 there exists a unique vector field $v \in \mathbb{R}^3$ that vanishes rapidly as $|x| \to \infty$ and with the property that

$$\omega = \operatorname{curl}u.$$

Thus, (i) clearly holds.

Next, consider the Poisson equation given by

$$\Delta\psi = \omega.$$

This Poisson equation can also be explicitly solved in terms of the Green's function:

$$\psi(x,t) = \frac{1}{4\pi}\int_\Omega \frac{\omega(y,t)}{|x-y|}\,dy. \tag{3.15}$$

Next, let the function $\psi^*, \bar{\psi}$ be such that

$$\psi^* = \operatorname{curl}\operatorname{curl}\psi, \qquad \bar{\psi} = \nabla\operatorname{div}\psi.$$

Since $\omega\colon [0,T) \to L^2(\Omega;\mathbb{R}^d)$ is smooth and vanishes rapidly as $|x| \to \infty$, it holds that

$$\psi^*(x,\cdot), \bar{\psi}(x,\cdot) = \mathcal{O}(|x|^{-3}) \qquad \text{for } |x| \gg 1.$$

Thus, $\psi^*\colon [0,T) \to L^2(\Omega;\mathbb{R}^d)$ and $\bar{\psi}\colon [0,T) \to L^2(\Omega;\mathbb{R}^d)$, and moreover by Equation (3.14) it holds that

$$\psi^* + \bar{\psi} = \omega.$$

Taking the L^2 inner product with $\bar{\psi}$ on both sides of the above equation, we obtain

$$(\bar{\psi},\bar{\psi}) = (\omega,\bar{\psi}) - (\psi^*,\bar{\psi}).$$

Now, observe that

$$\begin{aligned}(\omega,\bar{\psi}) &= \int_\Omega \omega \cdot (\nabla \operatorname{div} \psi)\, dx \\ &\underbrace{=}_{\substack{\text{Integration}\\ \text{by parts}}} - \int_\Omega \operatorname{div}\omega \operatorname{div}\psi\, dx \\ &\underbrace{\equiv}_{(3.14)} 0.\end{aligned}$$

Similarly, it also holds that

$$\begin{aligned}(\psi^*,\bar{\psi}) &= -\int_\Omega \operatorname{curl}\operatorname{curl}\psi \cdot \nabla \operatorname{div}\psi\, dx \\ &\underbrace{=}_{\substack{\text{Integration}\\ \text{by parts}}} \int_\Omega \operatorname{div}\big(\operatorname{curl}\operatorname{curl}\psi\big) \operatorname{div}\psi\, dx \\ &\underbrace{\equiv}_{(3.14)} 0.\end{aligned}$$

Hence, $(\bar{\psi},\bar{\psi}) = 0$, and therefore, $\bar{\psi} \equiv 0$. Thus, Equation (3.14) implies that

$$\operatorname{curl}(-\operatorname{curl}\psi) = \Delta\psi = \omega.$$

Now, let $u = -\operatorname{curl}\psi$. Then, clearly u satisfies Equation (3.13) and moreover, an explicit form for u can be derived from Equation (3.15) as

$$u(x,t) = \frac{1}{4\pi}\int_\Omega \frac{x-y}{|x-y|^3} \times \omega(y,t)\, dy. \tag{3.16}$$

The proof is thus complete. □

Theorem 3.4 enables us to re-write the unknown velocity field u that appears in Equation (3.14) in terms of the vorticity ω. Unfortunately, we must also eliminate the gradient of the unknown velocity field ∇u. Unfortunately, it is not possible to obtain a similar expression for ∇u using simple differentiation under the integral sign. Indeed, we observe that for all $y \in \mathbb{R}^d, y \neq x$, the term

$$\nabla_x \frac{x-y}{|x-y|^3},$$

is homogeneous of degree -3 and therefore, the singularity is not integrable on $\Omega = \mathbb{R}^3$. Hence, we must instead use distribution theory in order to compute ∇u.

We first require the following lemma [MB02, Proposition 2.17]:

Lemma 3.5 *Let $d > 1$ be an integer, let $K\colon \mathbb{R}^d \to \mathbb{R}$ be a locally integrable function homogeneous of degree $1-d$ such that K is smooth away from $x = 0$. Then for all $j \in \{1, \dots, d\}$, the distributional derivative ∂_{x_j} of K is the linear functional $\partial_{x_j} K$ with the property that for all $\phi \in C_0^\infty(\mathbb{R}^d)$ it holds that*

$$\begin{aligned}(\partial_{x_j} K, \phi) &= -(K, \partial_{x_j}\phi)\\ &= PV\int_{\mathbb{R}} \partial_{x_j} K\phi\, dx - c_j(\delta_0, \phi).\end{aligned}$$

Here, δ_0 is the Dirac distribution centred at zero, the constant c_j is given by

$$c_j = \int_{|x|=1} K(x) x_j\, ds(x),$$

and we have used the notation $PV\int_{\mathbb{R}^d}(\cdot)\, dx$ to denote the Cauchy principal-value *integral:*

$$PV\int_{\mathbb{R}^d} f\, dx = \lim_{\epsilon \searrow 0}\int_{|x|\geq\epsilon} f\, dx, \qquad \textit{for all } f \in L^1(\mathbb{R}^d).$$

Proof By definition of the distributional derivative, for all $j \in \{1, \dots, d\}$ and for all $\phi \in C_0^\infty(\mathbb{R}^d)$ it holds that

$$-(\partial_{x_j} K, \phi) = \int_{\mathbb{R}^d} K \partial_{x_j} \phi \, dx.$$

Since $K \in L^1_{loc}(\mathbb{R}^d)$, the dominated convergence theorem implies that for all $j \in \{1, \dots, d\}$ and for all $\phi \in C_0^\infty(\mathbb{R}^d)$ it holds that

$$\begin{aligned} \int_{\mathbb{R}^d} K \partial_{x_j} \phi \, dx = & \lim_{\epsilon \searrow 0} \int_{|x| \geq \epsilon} K \partial_{x_j} \phi \, dx \\ \underbrace{=}_{\substack{\text{Green's} \\ \text{theorem}}} & \lim_{\epsilon \searrow 0} \Big(\underbrace{- \int_{|x| \geq \epsilon} \partial_{x_j} K \phi \, dx}_{T_1} + \underbrace{\int_{|x| = \epsilon} K \phi \frac{x_j}{|x|} \, ds(x)}_{T_2} \Big). \end{aligned}$$

By definition, it holds that

$$\lim_{\epsilon \searrow 0} T_1 = - \lim_{\epsilon \searrow 0} \int_{|x| \geq \epsilon} \partial_{x_j} K \phi \, dx = -\mathrm{PV} \int_{\mathbb{R}} \partial_{x_j} K \phi \, dx.$$

Moreover, for every $\epsilon > 0$, we can introduce a change of variables $y = \frac{x}{\epsilon}$ in the term T_2 to obtain $ds(y) = \frac{1}{\epsilon^{d-1}} ds(x)$ and therefore

$$\begin{aligned} T_2 = \int_{|x| = \epsilon} K(x) \phi(x) \frac{x_j}{|x|} \, ds(x) = & \int_{|y|=1} K(\epsilon y) \phi(\epsilon y) y_j \epsilon^{d-1} \, ds(y) \\ \underbrace{=}_{\text{homogenity}} & \int_{|y|=1} \epsilon^{1-d} K(y) \phi(\epsilon y) y_j \epsilon^{d-1} \, ds(y) \\ = & \int_{|y|=1} K(y) \phi(\epsilon y) y_j \, ds(y). \end{aligned}$$

Finally, taking the limit $\epsilon \searrow 0$, we obtain

$$\begin{aligned} \lim_{\epsilon \searrow 0} T_2 &= \phi(0) \int_{|y|=1} K(y) y_j \, ds(y) \\ &= c_j(\delta_0, \phi). \end{aligned}$$

The proof is thus complete. □

We remark that the gradient ∇K is homogeneous of degree $-N$, has mean-value zero on the unit sphere and is consequently an example of a *singular integral operator* (SIO).

We can now use Lemma 3.5 in order to compute the gradient of the velocity field given by (3.16). Due to the tedious and lengthy nature of this computation, we skip the details and instead refer the reader to [MB02, Section 2.4.3]. Eventually, we arrive at the following expression:

$$\begin{aligned}[\nabla u(x,t)]h = &- \mathrm{PV} \int_\Omega \Bigg(\frac{1}{4\pi} \frac{\omega(y,t) \times h}{|x-y|^3} \\ &+ \frac{3}{4\pi} \frac{\big(\big((x-y) \times \omega(y,t)\big) \otimes (x-y)\big)h}{|x-y|^5} \Bigg) dy \\ &+ \frac{1}{3}\omega(x) \times h, \qquad\qquad \forall\ h \in \mathbb{R}^3. \end{aligned} \tag{3.17}$$

In conclusion, we obtain the following vorticity-stream function formulation for the 3-D incompressible Euler equation (3.2):

Lemma 3.6 *Let $T \in (0,\infty]$, let $d = 3$, let $\Omega = \mathbb{R}^d$ and let $u \colon \Omega \times [0,T) \to \mathbb{R}^d$ be a velocity vector field with the property that u vanishes rapidly as $|x| \to \infty$. Then the velocity-pressure formulation of the 3-D incompressible Euler equations* (3.2) *is equivalent to the following vorticity-stream function formulation:*

$$\begin{aligned} \frac{D\omega}{dt} = \partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u \quad &\textit{for all } (x,t) \in \mathbb{R}^d \times (0,T), \\ \omega(x,0) = \omega_0(x) \quad &\textit{for all } x \in \mathbb{R}^d, \end{aligned} \tag{3.18}$$

where the velocity u is given by

$$u(x,t) = \frac{1}{4\pi} \int_\Omega \frac{x-y}{|x-y|^3} \times \omega(y,t)\, dy,$$

and the gradient of the velocity ∇u is defined in the sense of distributions as

$$\begin{aligned}[\nabla u(x,t)]h = &- PV \int_\Omega \Bigg(\frac{1}{4\pi} \frac{\omega(y,t) \times h}{|x-y|^3} \\ &+ \frac{3}{4\pi} \frac{\big(\big((x-y) \times \omega(y,t)\big) \otimes (x-y)\big)h}{|x-y|^5} \Bigg) dy \\ &+ \frac{1}{3}\omega(x) \times h, \qquad\qquad \forall\ h \in \mathbb{R}^3. \end{aligned}$$

Moreover, the pressure field p can be obtained by solving the Poisson equation (3.3)*.*

3.2 Lagrangian Reformulation of the Incompressible Euler Equations

Consider the Incompressible Euler equations (3.2). In Section 3.1 we showed that this velocity-pressure formulation of the incompressible Euler equations is equivalent to the vorticity-stream function formulation (3.12).

We now introduce a third equivalent reformulation of the Euler equations. Recall from Section 1.2 and Section 3.1.3 that we can introduce Lagrangian trajectories as functions $\mathcal{X}\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ that satisfy the following initial value problem:

$$\begin{aligned} \frac{d\mathcal{X}}{dt}(a,t) &= u\big(\mathcal{X}(a,t),t\big), \\ \mathcal{X}(a,0) &= a. \end{aligned} \tag{3.19}$$

Therefore, given an initial vorticity field $\omega_0\colon \Omega \to \mathbb{R}^n$, where $n \in \{1,3\}$, it is of interest to calculate the rate of change of vorticity along these particle trajectories:

$$\frac{d}{dt}\omega\big(\mathcal{X}(a,t),t\big).$$

To this end, we make use of the following lemma.

Lemma 3.7 *Let $T \in (0,\infty]$, let $N \in \mathbb{N}$, let $d \in \{2,3\}$, let $\Omega = \mathbb{R}^d$, let $u\colon \Omega \times [0,T) \to \mathbb{R}^d$ be a smooth velocity vector field with associated flow map $\mathcal{X}$ defined by Equation* (3.19) *and let $h\colon \Omega \times [0,T) \to \mathbb{R}^N$ be a smooth vector field. Then*

$$\frac{Dh}{dt} := \partial_t h + (u\cdot\nabla)h = (h\cdot\nabla)u \tag{3.20}$$

$$\iff h\big(\mathcal{X}(a,t),t\big) = \nabla_a\big(\mathcal{X}(a,t)\big)h_0(a), \tag{3.21}$$

where we have used the notation that $h_0(a) := h(a,0)$ for all $a \in \Omega$.

Proof Consider Equation (3.19) and differentiate both sides with respect to the spatial variable. It follows from the chain rule that the following matrix equation holds:

$$\frac{d}{dt}\nabla_a \mathcal{X}(a,t) = \nabla_a u\big(\mathcal{X}(a,t),t\big) : \nabla_a \mathcal{X}(a,t). \tag{3.22}$$

Using the notation that $\nabla_a u\big(\mathcal{X}(a,t),t\big) := (\nabla u)|_{\mathcal{X}(a,t)}$ and multiplying both sides of Equation (3.22) with the vector $h_0(a)$ we obtain

$$\frac{d}{dt}\nabla_a \mathcal{X}(a,t) h_0(a) = (\nabla u)|_{\mathcal{X}(a,t)} \nabla_a \mathcal{X}(a,t) h_0(a). \tag{3.23}$$

Moreover, if h satisfies Equation (3.20), it holds that

$$\frac{\mathrm{D}h}{\mathrm{dt}} = h \cdot \nabla u,$$

and therefore it follows that

$$\frac{d}{dt} h\big(\mathcal{X}(a,t),t\big) = (\nabla u)|_{\mathcal{X}(a,t)} h\big(\mathcal{X}(a,t),a\big). \tag{3.24}$$

Thus, both h and $\nabla_a\big(\mathcal{X}(a,t)\big)h_0(a)$ satisfy the same linear ordinary differential equation (3.23) and (3.24) with the same initial datum $h_0(a)$. It therefore follows from uniqueness of solutions to ODEs that for all $a \in \Omega$ and for all $t \in [0,T)$ it holds that $h\big(\mathcal{X}(a,t),t\big) = \nabla_a \mathcal{X}(a,t) h_0(a)$. Since these last step are reversible, the proof is complete. □

Some remarks are now in order.

Remark 3.8 *Given a 3-D flow map $\mathcal{X}$, the gradient $\nabla_a \mathcal{X}(a,t)$ is a matrix defined as*

$$\nabla_a \mathcal{X}(a,t) = \begin{bmatrix} \frac{\mathcal{X}_1(a,t)}{da_1} & \frac{\mathcal{X}_1(a,t)}{da_2} & \frac{\mathcal{X}_1(a,t)}{da_3} \\ \frac{\mathcal{X}_2(a,t)}{da_1} & \frac{\mathcal{X}_2(a,t)}{da_2} & \frac{\mathcal{X}_2(a,t)}{da_3} \\ \frac{\mathcal{X}_3(a,t)}{da_1} & \frac{\mathcal{X}_3(a,t)}{da_2} & \frac{\mathcal{X}_3(a,t)}{da_3} \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{X}^1_{a_1} & \mathcal{X}^1_{a_2} & \mathcal{X}^1_{a_3} \\ \mathcal{X}^2_{a_1} & \mathcal{X}^2_{a_2} & \mathcal{X}^2_{a_3} \\ \mathcal{X}^3_{a_1} & \mathcal{X}^3_{a_2} & \mathcal{X}^3_{a_3} \end{bmatrix},$$

where we have used the notation that $\mathcal{X} := (\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3)$ and $a := (a_1, a_2, a_3)$.

Note that since div u=0*, it can be shown that the determinant of the above matrix* $\det\nabla_a\mathcal{X}(a,t) \equiv 1$ *for all* $a \in \Omega$ *and for all* $t \in [0,T)$*. We leave this proof as an exercise.*

Moreover, without loss of generality, we may assume that the above matrix has three complex eigenvalues denoted by λ, λ^{-1} *and* 1*, with* $|\lambda| \geq 1$*.*

Remark 3.9 *In the context of Lemma 3.7, we may, in particular, take* $h = \omega$*. The vorticity* ω *satisfies Equation* (3.20) *and therefore also satisfies Equation* (3.21)*. It follows that for all* $a \in \Omega$ *and for all* $t \in [0,T)$ *the vorticity satisfies the equation*

$$\omega\big(\mathcal{X}(a,t),t\big) = \nabla_a\mathcal{X}(a,t)\omega_0(a). \tag{3.25}$$

Hence, along streamlines $\mathcal{X}(a,t)$*, the vorticity is stretched by a factor* $\nabla_a\mathcal{X}(a,t)$*. In view of Remark 3.8, when* ω_0 *is aligned with the complex eigenvector associated with the eigenvalue* λ*, the vorticity is amplified by the flow.*

Remark 3.10 *For 2-D flows, the situation is much simpler. Indeed, we observe that in two-spatial dimensions, the initial vorticity is given by* $\omega(a) = (0,0,\omega_3(a))$ *and the gradient of the flow map* $\mathcal{X}$ *is given by*

$$\nabla_a\mathcal{X}(a,t) = \begin{bmatrix} \mathcal{X}^1_{a_1} & \mathcal{X}^1_{a_2} & 0 \\ \mathcal{X}^2_{a_1} & \mathcal{X}^2_{a_2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the vorticity evolution equation (3.25) *simply reduces to*

$$\omega\big(\mathcal{X}(a,t),t\big) = \omega_0(a). \tag{3.26}$$

Thus, for 2-D flows, vorticity is preserved along the streamlines.

Equation (3.25) lies at the heart of the so-called *Lagrangian reformulation* of the incompressible Euler equations (3.2). Indeed, let us consider the case of three spatial dimensions, i.e., $d = 3$. We recall from Section 3.1.3 that the velocity field u is given by the integral equation

$$u(x,t) = \int_\Omega K_3(x-y)\omega(y,t)\,dy,$$

with kernel K_3 given by

$$K_3(x)h = \frac{1}{4\pi}\frac{x \times h}{|x|^3}, \qquad \forall\ h \in \mathbb{R}^3. \tag{3.27}$$

Let us now rewrite this expression for the velocity by introducing the change of variables $y = \mathcal{X}(a', t)$. We thus obtain

$$u(x,t) = \int_\Omega K_3\big(x - \mathcal{X}(a',t)\big)\omega\big(\mathcal{X}(a',t),t\big)\, da',$$

and Equation (3.25) therefore implies that

$$u(x,t) = \int_\Omega K_3\big(x - \mathcal{X}(a',t)\big)\nabla_a \mathcal{X}(a',t)\omega_0(a')\, da'.$$

Hence, we arrive at the following Lagrangian reformulation of the 3-D incompressible Euler equations (3.2):

$$\begin{aligned} \frac{d\mathcal{X}(a,t)}{dt} &= \int_\Omega K_3\big(\mathcal{X}(a,t) - \mathcal{X}(a',t)\big)\nabla_a \mathcal{X}(a',t)\omega_0(a')\, da', \\ \mathcal{X}(a,0) &= a. \end{aligned} \tag{3.28}$$

Equation (3.28) is an integro-differential equation for the Lagrangian trajectories generated by the flow-map $\mathcal{X}$ and is completely equivalent to the velocity-pressure formulation of the Euler equations (3.2). A detailed proof of this fact can, e.g., be found in [MB02, Proposition 2.23]

3.2.1 Classical Solutions to the Incompressible Euler Equations

Throughout this section, unless stated otherwise, we restrict out attention to the case of three-dimensional flows. We have previously shown in Section 3.2 that the Lagrangian reformulation of the 3-D incompressible Euler equation is given by Equation (3.28):

$$\begin{aligned} \frac{d\mathcal{X}(a,t)}{dt} &= \int_{\mathbb{R}^3} K_3\big(\mathcal{X}(a,t) - \mathcal{X}(a',t)\big)\nabla_a \mathcal{X}(a',t)\omega_0(a')\, da' \\ \mathcal{X}(a,0) &= a. \end{aligned}$$

Observe that Equation (3.28) can be rewritten as a non-linear ordinary differential equation on an infinite-dimensional space as

$$\begin{aligned} \frac{d\mathcal{X}(a,t)}{dt} &= F\big(\mathcal{X}(a,t),t\big), \\ \mathcal{X}(a,0) &= a, \end{aligned} \tag{3.29}$$

where the non-linear mapping F is given by

$$F\big(\mathcal{X}(a,t)\big) = \int_{\mathbb{R}^3} K_3\big(\mathcal{X}(a,t) - \mathcal{X}(a',t)\big)\nabla_a \mathcal{X}(a',t)\omega_0(a')\,da'.$$

Our goal is now to show that there exists some infinite-dimensional Banach Space $\mathbb{B}$ such that F is a locally Lipschitz continuous function on $\mathbb{B}$ or some subset of $\mathbb{B}$. This will allow us to appeal to classical existence and uniqueness theory for ODEs to prove the existence of a local in time solution of Equation (3.29). In particular, we intend to use the following version of the Picard existence theorem for Banach spaces:

Theorem 3.11 *Let $\mathbb{B}$ be a Banach space, let $O \subseteq \mathbb{B}$ be an open set and let F be a non-linear operator satisfying the following conditions:*

- *$F\colon O \to \mathbb{B}$;*
- *F is locally Lipschitz continuous on O, i.e., for every $X \in O$, there exists some constant $L > 0$ and an open set $V_X \subset O$ such that for all $\bar{X}, \hat{X} \in V_X$ it holds that*

$$\|F(\bar{X}) - F(\hat{X})\|_{\mathbb{B}} \leq L\|\bar{X} - \hat{X}\|_{\mathbb{B}}.$$

Then for any $X_0 \in O$, there exists a time $T > 0$ such that the ODE

$$\begin{aligned} \frac{d\mathcal{X}(t)}{dt} &= F(\mathcal{X}(t)), \\ \mathcal{X}|_{t=0} &= X_0, \end{aligned} \tag{3.30}$$

has a unique local solution $X \in C^1\big((-T,T);O\big)$.

Proof This theorem can, for example, be found in [MB02, Theorem 3.1]. The reader is referred to [Har82] for a proof. □

Clearly, in order to apply Theorem 3.11 to Equation (3.29), we must first choose an appropriate Banach space $\mathbb{B}$ and an open set $O \subset \mathbb{B}$ such that conditions of the theorem are satisfied. In particular, for any $t > 0$, the open set O must contain all maps $\mathcal{X}(\cdot,t)\colon \mathbb{R}^3 \to \mathbb{R}^3$ that are injective and surjective, including the identity map. Moreover, the Banach space $\mathbb{B}$ must be such that F is locally Lipschitz continuous on $O \subseteq \mathbb{B}$.

The Choice of the Banach space $\mathbb{B}$:

First, note that since the non-linear mapping F appearing in Equation (3.29) explicitly contains the gradient term $\nabla_a \mathcal{X}$, we may naively assume that it suffices to set the Banach space $\mathbb{B} = C^1(\mathbb{R}^3; \mathbb{R}^3)$, i.e., the space of all bounded and continuously differentiable functions. Unfortunately, this choice of $\mathbb{B}$ does not work for two reasons. First, observe that if $\mathcal{X} \in C^1(\mathbb{R}^3; \mathbb{R}^3)$ is injective and surjective, then $\mathcal{X}$ cannot be bounded. Second, note that the non-linear mapping F contains the gradient $\nabla_a \mathcal{X}$, which, as discussed in Section 3.1.3, is a singular integral operator (SIO). Such SIOs do not map the class of bounded functions to itself and instead map the set of L^∞ functions to the larger set of functions with *bounded mean oscillation* (BMO). This implies that the non-linear mapping F is not bijective on $\mathbb{B} = C^1(\mathbb{R}^3; \mathbb{R}^3)$.

It turns out that the correct choice is a class of *Hölder* continuous functions. Indeed, we define the set $\mathbb{B}$ as

$$\mathbb{B} = \left\{ \mathcal{X} \colon \mathbb{R}^3 \to \mathbb{R}^3 \text{ and } |\mathcal{X}|_{1,\gamma} < \infty \quad \text{for } \gamma \in (0,1) \right\}, \tag{3.31}$$

where the norm $|\cdot|_{1,\gamma}$ is defined as

$$|\mathcal{X}|_{1,\gamma} = |\mathcal{X}(0)| + |\nabla_a \mathcal{X}|_0 + |\nabla_a \mathcal{X}|_\gamma,$$

and we have used the usual notation that

$$|\mathcal{X}(a)|_0 = \sup_{a \in \mathbb{R}^3} |\mathcal{X}(a)|,$$

and

$$|\mathcal{X}(a)|_\gamma = \sup_{\substack{a,a' \in \mathbb{R}^3 \\ a \neq a'}} \frac{|\mathcal{X}(a) - \mathcal{X}(a')|}{|a - a'|^\gamma},$$

define the supremum norm and the Hölder norm respectively.

It can be shown that the space $\mathbb{B}$ defined above is a complete, normed-vector space and thus indeed a Banach space. Furthermore, as we discuss below, this choice of the Banach space $\mathbb{B}$ satisfies our needs and will allow us us to apply Theorem 3.11 to obtain local in time existence of solutions to Equation (3.29).

The Choice of the Open Subset $O \subseteq \mathbb{B}$:

In order to apply the Picard existence Theorem 3.11, we must select an open set $O \subseteq \mathbb{B}$ such that the identity map, which is the initial condition of ODE (3.29), is contained in O and moreover, the non-linear mapping F is locally Lipschitz continuous on O.

As mentioned in Remark 3.8, the incompressibility constraint $\operatorname{div} u = 0$ implies that $\nabla_a \mathcal{X} \equiv 1$ for all $a \in \mathbb{R}^3$. Unfortunately, this condition is too restrictive as it defines a hyperplane of functions in the Banach space $\mathbb{B}$, whereas we would like to find an open set. Thus, we instead define for every $M > 0$ the set O_M given by

$$O_M = \left\{ \mathcal{X} \in \mathbb{B} \colon \inf_{a \in \mathbb{R}^3} \det \nabla_a \mathcal{X}(a) > \frac{1}{2} \text{ and } |X|_{1,\gamma} < M \quad \text{for } \gamma \in (0,1) \right\}.$$

It can then be shown (see, e.g., [MB02, Proposition 4.1]) that for any $M > 0$ and for any $\gamma \in (0,1)$, the set O_M is non-empty, open and consists of bijective mappings (homeomorphisms) of $\mathbb{R}^3$ onto $\mathbb{R}^3$.

We can now apply the Picard existence theorem 3.11 to Equation (3.29). We therefore obtain the following local existence and uniqueness theorem (due to Leon Lichtenstein) for solutions to the initial value problem (3.29).

Theorem 3.12 *Let the kernel K_3 be defined by Equation* (3.27)*, let $\gamma \in (0,1)$ and let $\omega_0 \in C^{0,\gamma}(\mathbb{R}^3;\mathbb{R}^3)$ be a compactly supported function. Then, for any $M > 0$, there exists some time $T(M) > 0$ and a unique solution*

$$\mathcal{X} \in C^1\big((-T(M), T(M)); O_M\big)$$

to the initial value problem

$$\begin{aligned} \frac{d\mathcal{X}(a,t)}{dt} &= \int_\Omega K_3\big(\mathcal{X}(a,t) - \mathcal{X}(a',t)\big) \nabla_a \mathcal{X}(a',t) \omega_0(a')\, da', \\ \mathcal{X}(a,0) &= a, \end{aligned}$$

and by extension to the incompressible Euler equations (3.2)*.*

Proof (Sketch) Theorem 3.12 and its proof can be found in [MB02, Theorem 4.2]. We present here a sketch of the proof.

Our aim is to demonstrate that the mapping $F \colon O_M \to \mathbb{B}$ defined by (3.29) is both bounded and locally Lipschitz continuous. To this end, for clarity of exposition we suppress the time-dependence of the function F and we write for each $\mathcal{X} \in O_M$

$$F(\mathcal{X}) = \int_{\mathbb{R}^3} K_3\big(\mathcal{X}(a) - \mathcal{X}(\bar{a})\big)\nabla_a \mathcal{X}(\bar{a})\omega_0(\bar{a})\, d\bar{a}.$$

Furthermore, we introduce a change of variables by defining

$$\begin{aligned} x &= \mathcal{X}(a), & \bar{x} &= \mathcal{X}(\bar{a}), \\ a &= \mathcal{X}^{-1}(x), & \bar{a} &= \mathcal{X}^{-1}(\bar{x}). \end{aligned}$$

The fact that the flow-map $\mathcal{X} \in O_M$ is a homeomorphism implies that this change of variables is well-defined. It therefore follows that for each $\mathcal{X} \in O_M$ it holds that

$$F(\mathcal{X}) = \int_{\mathbb{R}^3} K_3\big(\mathcal{X}(\mathcal{X}^{-1}(x)) - \bar{x}\big)\nabla_a \mathcal{X}(\mathcal{X}^{-1}(\bar{x}))\omega_0(\mathcal{X}^{-1}(\bar{x}))\det\big(\nabla_x \mathcal{X}^{-1}(\bar{x})\big)\, d\bar{x}.$$

Thus, using $\circ$ to denote function decomposition, we may write the mapping F in the compact form

$$F(\mathcal{X}) = (K_3 f) \circ \mathcal{X}^{-1},$$

where

$$K_3 f(x) = \int_{\mathbb{R}^3} K_3(x - \bar{x}) f(\bar{x})\, d\bar{x},$$

and

$$f(\bar{x}) = \nabla_a \mathcal{X}(\bar{a})\omega_0(\bar{a})\det\big(\nabla_x \mathcal{X}^{-1}(\bar{x})\big).$$

A straightforward calculus identity then yields that there exists some constant $C > 0$ such that for all $\mathcal{X} \in O_M$ and $\gamma \in (0,1)$ it holds that

$$|F(\mathcal{X})|1,\gamma \leq |K_3 f|_{1,\gamma} |\mathcal{X}^{-1}|_{1,\gamma} \leq C |K_3 f|_{1,\gamma} |\mathcal{X}|^2_{1,\gamma}.$$

Hence, it remains to bound the term $|K_3 f|_{1,\gamma}$.

We recall that $K_3 f$ is a solution to the Poisson equation

$$\Delta(K_3 f) = f,$$

and therefore by the Schauder estimates of potential theory, for every $\alpha \in (0,1)$ there exists some constant $\tilde{C} > 0$ such that

$$\|K_3 f\|_{C^{2,\alpha}} \leq \tilde{C}\|f\|_{0,\alpha},$$

and a slight modification of this argument yields that for for every $\alpha \in (0,1)$ there exists some constant $\bar{C} > 0$ such that

$$|K_3 f|_{1,\alpha} \leq \bar{C}\|f\|_{0,\alpha}. \tag{3.32}$$

It therefore suffices to bound the term $\|f\|_{0,\alpha}$. By definition, for all $\alpha \in (0,1)$ it holds that

$$\|f\|_{0,\alpha} = \|\nabla_a \mathcal{X}(\mathcal{X}^{-1})\omega_0(\mathcal{X}^{-1})\det(\nabla_x \mathcal{X}^{-1})\|_{0,\alpha}.$$

Two simple calculus identities imply for every $\alpha \in (0,1)$, the existence of constants $C_1, C_2 > 0$ such that

$$\begin{aligned}\|f \circ \mathcal{X}^{-1}\|_{0,\alpha} &\leq \|f\|_{0,\alpha}\Big(1 + C\|\mathcal{X}\|_{1,\alpha}^{\alpha(2d-1)}\Big),\\ \|\nabla_a \mathcal{X}^{-1}\|_{0,\alpha} &\leq C|\mathcal{X}|_{1,\alpha}^{2d-1}.\end{aligned}$$

and hence for every $\alpha \in (0,1)$ there exists a constant $C > 0$ such that

$$\|f\|_{0,\alpha} \leq \|\nabla_a \mathcal{X}\|_{0,\alpha}\|\omega_0\|_{0,\alpha}\Big(1 + C\|\mathcal{X}\|_{1,\alpha}^{\alpha(2d-1)}\Big)|\mathcal{X}|_{1,\alpha}^{2d-1}.$$

Since each term in the above inequality is finite, we conclude that for all $\mathcal{X} \in O_M$ and $\gamma \in (0,1)$ it holds that

$$|F(\mathcal{X})|_{1,\gamma} < \infty.$$

Next, we must prove that the mapping $F\colon O_M \to B$ is locally Lipschitz continuous. Note that if the Fréchet derivative $F'(\mathcal{X})$ is bounded for all $\mathcal{X} \in O_M$, then the mean-value theorem implies that for all $\tilde{\mathcal{X}}, \bar{\mathcal{X}} \in O_M$ and all $\gamma \in (0,1)$ it holds that

$$\begin{aligned}|F(\tilde{\mathcal{X}}) - F(\bar{\mathcal{X}})|_{1,\gamma} &= \left|\int_0^1 \frac{d}{d\epsilon} F\big(\tilde{\mathcal{X}} + \epsilon(\bar{\mathcal{X}} - \tilde{\mathcal{X}})\big)\, d\epsilon\right|_{1,\gamma}\\ &\leq \int_0^1 \big|F'\big(\tilde{\mathcal{X}} + \epsilon(\bar{\mathcal{X}} - \tilde{\mathcal{X}})\big)\big|_{1,\gamma}\, d\epsilon |\tilde{\mathcal{X}} - \bar{\mathcal{X}}|_{1,\gamma},\end{aligned}$$

and thus F is indeed locally Lipschitz continuous on O_M. Therefore, it is sufficient to show that the Fréchet derivative F' is a bounded linear operator on O_M. To this end, observe that for all $\mathcal{X}, Y \in O_M$ it holds that

$$\begin{aligned} F'(\mathcal{X})Y &= \frac{d}{d\epsilon} F(\mathcal{X} + \epsilon Y)|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \int_{\mathbb{R}^3} K_3\Big(\mathcal{X}(a) - \mathcal{X}(\bar{a}) + \epsilon\big(Y(a) - Y(\bar{a})\big)\Big) \\ &\quad \cdot \nabla_a\big(\mathcal{X}(\bar{a}) + \epsilon Y(\bar{a})\big)\omega_0(\bar{a})\, d\bar{a}\Big|_{\epsilon=0} \\ &= \underbrace{\int_{\mathbb{R}^3} K_3\big(\mathcal{X}(a) - \mathcal{X}(\bar{a})\big)\nabla_a Y(\bar{a})\omega_0(\bar{a})\, d\bar{a}}_{:=T_1} \\ &\quad + \underbrace{\int_{\mathbb{R}^3} \nabla K_3\big(\mathcal{X}(a) - \mathcal{X}(\bar{a})\big)\big(Y(a) - Y(\bar{a})\big)\nabla_a \mathcal{X}(\bar{a})\omega_0(\bar{a})\, d\bar{a}}_{:=T_2}. \end{aligned}$$

We must now estimate each term T_1, T_2 separately. First, we observe that the term T_1 can be written as

$$T_1 = \big(K_3 \nabla_a Y \omega_0\big) \circ \mathcal{X}^{-1},$$

and therefore can be estimated using the potential theory estimate (3.32). Indeed, we obtain that for any $\gamma \in (0,1)$ there exists some constant $C > 0$ such that for all $\mathcal{X}, Y \in O_M$ it holds that

$$|T_1|_{1,\gamma} \leq C\|\omega_0\|_{0,\gamma}|Y|_{1,\gamma}$$

The term T_2 can similarly be estimated using results from potential theory. We omit this proof due since it is significantly more technical but a detailed argument can, for example, be found in [MB02, Appendix]. We eventually obtain that for any $\gamma \in (0,1)$ there exists some constant $\tilde{C} > 0$ such that for all $\mathcal{X}, Y \in O_M$ it holds that

$$|T_2|_{1,\gamma} \leq \tilde{C}\|\omega_0\|_{0,\gamma}|Y|_{1,\gamma}$$

We therefore conclude that for any $\gamma \in (0,1)$ there exists some constant $\bar{C} > 0$ such that for all $\mathcal{X}, Y \in O_M$ it holds that

$$|F'(\mathcal{X})Y|_{1,\gamma} \leq C\|\omega_0\|_{0,\gamma}|Y|_{1,\gamma}.$$

Thus the Fréchet derivative $F' \colon O_M \to B$ is a bounded linear operator and therefore F is locally Lipschitz continuous. The proof is thus complete. □

3.2.2 Global Existence of Classical Solutions

Throughout this section, unless stated otherwise, we restrict out attention to the case of three-dimensional flows. In Section 3.2.1, we established the existence of a local unique solution to the Lagrangian formulation of the 3-D incompressible Euler equations (3.28). We now discuss the existence of *global* solutions to the IVP (3.28).

As in standard ODE theory, we can obtain global in time existence of solutions if there is no blow-up and no continuation of solutions outside the open set of local existence. We shall make use of the following abstract theorem:

Theorem 3.13 (Continuation Theorem) *Let $T \in (0, \infty]$, let $\mathbb{B}$ be a Banach space, let $O \subset \mathbb{B}$ be an open set, let $F\colon O \to \mathbb{B}$ be a locally Lipschitz continuous mapping and let $\mathcal{X} \in C^1\big([0,T]; O\big)$ be the unique solution of the following initial value problem involving the autonomous ODE:*

$$\frac{d\mathcal{X}}{dt} = F(\mathcal{X}),$$

$$\mathcal{X}|_{t=0} = \mathcal{X}_0 \in O.$$

Then one of the following must hold:

1. *$\mathcal{X}$ is a global in time solution, i.e., $T = \infty$;*
2. *$T < \infty$ and the solution $\mathcal{X}(t)$ leaves the open set O as $t \nearrow T$.*

Proof A proof of Theorem 3.13 can be found in any standard text on ODE theory such as, for example, [LL72, Page 161]. □

We can now apply Theorem 3.13 to the IVP (3.29) to obtain the following celebrated result, which is due to Beale, Kato and Majda [BKM84]:

Theorem 3.14 (Beale-Kato-Majda) *Let $\gamma \in (0,1)$, let $\omega_0\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a compactly supported function such that $\|\omega_0\|_{0,\gamma} < \infty$, let ω be a solution to the vorticity-stream function formulation of the incompressible Euler equations* (3.18) *with initial datum ω_0 on some time interval and let $|\omega(\cdot, s)|_0$ denote that C_0-norm at time s of the solution ω.*

1. *Suppose that for every $T > 0$, there exists a constant $M_1 > 0$ such that the vorticity ω satisfies the bound*

$$\int_0^T |\omega(\cdot, s)|_0 \, ds \le M_1.$$

Then for any $T > 0$ *there exists a constant* $M > 0$ *such that the solution to the Lagrangian formulation of the incompressible Euler equations* (3.29) $\mathcal{X} \in C^1\big([0,T]; O_M\big)$*. In other words, the solution* $\mathcal{X}$ *exists globally in time.*

2. *Suppose that for any* $M > 0$ *there is a finite maximal time* $T(M) > 0$ *of existence of solutions* $\mathcal{X} \in C^1\big([0,T(M)]; O_M\big)$ *to the Lagrangian formulation of the incompressible Euler equations* (3.29) *and further that* $\lim_{M\to\infty} T(M) = T^\infty < \infty$*. Then it holds that*

$$\lim_{t \nearrow T^*} \int_0^t |\omega(\cdot, s)|_0 \, ds = \infty.$$

In other words, either the time-integrated vorticity blows up or there is a global in time classical solution to the incompressible Euler equations.

Proof (Sketch) A detailed proof of Theorem 3.14 can, for example, be found in [MB02, Theorem 4.3]. We present here a sketch of the proof.

Our aim is to apply Theorem 3.13 to the IVP (3.29) involving the Lagranigian formulation of the incompressible Euler equations. To this end, we must show that for any $\gamma \in (0,1)$ there exists some constant $M > 0$ such that the solution $\mathcal{X}$ to (3.29) satisfies the bound

$$|\mathcal{X}|_{1,\gamma} \leq M.$$

We proceed in two steps.

Step 1:

Let $T > 0$. We first show that for any $\gamma \in (0,1)$, it holds that $|\mathcal{X}(\cdot,t)|_{1,\gamma}$ is a priori bounded on the time interval [0, T] provided that for all $t \in [0,T]$ there is an a priori bound on

$$\int_0^t |\nabla u(\cdot, s)|_0 \, ds.$$

We recall that the solution $\mathcal{X}$ satisfies the ordinary differential equation

$$\begin{aligned} \frac{d\mathcal{X}(a,t)}{dt} &= \int_{\mathbb{R}^3} K_3\big(\mathcal{X}(a,t) - \mathcal{X}(\bar{a},t)\big) \nabla_a \mathcal{X}(\bar{a},t)\omega_0(\bar{a}) \, d\bar{a} \\ &= u(\mathcal{X}(a,t),t). \end{aligned} \tag{3.33}$$

Differentiating both sides of the above equation, we obtain

$$\frac{d}{dt}\nabla_a \mathcal{X}(a,t) = \nabla u\big(\mathcal{X}(a,t),t\big)\nabla_a \mathcal{X}(a,t). \tag{3.34}$$

Schauder estimates from potential theory then imply that for each $t \in [0,T]$ there exists some constant $\bar{C} > 0$ such that

$$|u(\cdot,t)|_0 \leq \bar{C}R(t)|\omega(\cdot,s)|_0,$$

where $R(t)^3$ is the measure of the support of the vorticity ω at time t. By assumption, the initial vorticity is compactly supported and the flow map $\mathcal{X}$ is volume preserving. It follows that $R(t)$ is independent of t and is thus a constant. Hence, we obtain that there exists some constant $C > 0$ such that for all $t \in [0,T]$ it holds that

$$|u(\cdot,t)|_0 \leq C|\omega(\cdot,s)|_0. \tag{3.35}$$

The mean-value theorem implies that for each $t \in [0,T]$ it holds that

$$\begin{aligned}\mathcal{X}(0,t) - \underbrace{\mathcal{X}(0,0)}_{=0} &= \int_0^t \frac{d\mathcal{X}}{dt}(0,s)\,ds\\ &= \int_0^t u\big(\mathcal{X}(0,s),s\big)\,ds,\end{aligned}$$

and therefore there exists a constant $C > 0$ such that for all $t \in [0,T]$ it holds that

$$|\mathcal{X}(0,t)| \leq \int_0^t |u(\cdot,s)|_0\,ds \leq C\int_0^t |\omega(\cdot,s)|_0\,ds.$$

Note that Equation (3.34) implies that for each $t \in [0,T]$ it holds that

$$\frac{d}{dt}|\nabla_a \mathcal{X}(\cdot,t)|_0 \leq |\nabla u(\cdot,t)|_0 |\nabla_a \mathcal{X}(\cdot,t)|_0,$$

and thus, using Gronwall's inequality, we obtain that for each $t \in [0,T]$ it holds that

$$|\nabla_a \mathcal{X}(\cdot,t)|_0 \leq \exp\Big(\int_0^t |\nabla u(\cdot,s)|_0\,ds\Big). \tag{3.36}$$

Next, let $\gamma \in (0,1)$. Then Equation (3.34) implies that for each $t \in [0,T]$ it holds that

$$\begin{aligned}\frac{d}{dt}|\nabla_a \mathcal{X}(\cdot,t)|_\gamma \leq& |\nabla u\big(\mathcal{X}(\cdot,t),t\big)|_\gamma |\nabla_a \mathcal{X}(\cdot,t)|_0 + |\nabla u(\cdot,t)|_0 |\nabla_a \mathcal{X}(\cdot,t)|_\gamma \\ \leq& |\nabla u(\cdot,t)|_\gamma |\nabla_a \mathcal{X}(\cdot,t)|_0^{1+\gamma} + |\nabla u(\cdot,t)|_0 |\nabla_a \mathcal{X}(\cdot,t)|_\gamma,\end{aligned}$$

where the second inequality follows from the following simple calculus identity for smooth functions $f\colon \mathbb{R}^3 \to \mathbb{R}^3$:

$$|f \circ \mathcal{X}|_\gamma \leq |f|_\gamma |\nabla_a \mathcal{X}|_0^\gamma.$$

Inequality (3.36) can be used together with the following Schauder estimate: there exists some constant $C > 0$ such that for all $t \in [0,T]$ it holds that

$$|u(\cdot,t)|_\gamma \leq C|\omega(\cdot,s)|_\gamma,$$

to obtain that for all $t \in [0,T]$ it holds that

$$\begin{aligned}\frac{d}{dt}|\nabla_a \mathcal{X}(\cdot,t)|_\gamma \leq& C|\omega(\cdot,t)|_\gamma \exp\left((1+\gamma)\int_0^t |\nabla u(\cdot,s)|_0 \, ds\right) \\ &+ |\nabla u(\cdot,t)|_0 |\nabla_a \mathcal{X}(\cdot,t)|_\gamma.\end{aligned} \tag{3.37}$$

The next step is so estimate the vorticity term $|\omega(\cdot,t)|_\gamma$ in terms of $|\nabla v(\cdot,t)|_0$. We require the following lemma:

Lemma 3.15 *Let $\gamma \in (0,1)$, let $T > 0$, let $\omega_0 \in C^{0,\gamma}(\mathbb{R}^3;\mathbb{R}^3)$, let the function $\omega\colon \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ satisfy the vorticity-stream function formulation of the 3-D incompressible Euler equations given by*

$$\frac{d\omega}{dt} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u,$$

and let $\mathcal{X}\colon \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ be the flow-map associated with the velocity field u that satisfies Equations (3.33) *and* (3.34)*. Then there exists some constant $C_0 > 0$ such that for all $t \in [0,T]$ it holds that*

$$|\omega(\cdot,t)|_\gamma \leq |\omega_0|_\gamma \exp\left((C_0+\gamma)\int_0^t |\nabla u(\cdot,s)|_0 \, ds\right).$$

Proof This lemma and its proof can, for example be found in [MB02, Lemma 4.8]. □

We can now apply Lemma 3.15 to Inequality (3.37) to obtain the existence of constants $C_0, C_1 > 0$ such that for all $t \in [0,T]$ it holds that

$$\frac{d}{dt}|\nabla_a \mathcal{X}(\cdot,t)|_\gamma \leq C_0|\omega_0|_\gamma \exp\Big(C_1 \int_0^t |\nabla u(\cdot,s)|_0\, ds\Big) + |\nabla u(\cdot,t)|_0 |\nabla_a \mathcal{X}(\cdot,t)|_\gamma.$$

Gronwall's lemma then implies that there exist constants $C_0, C_1 > 0$ such that for all $t \in [0,T]$ it holds that

$$|\nabla_a \mathcal{X}(\cdot,t)|_\gamma \leq C_0|\omega_0|_0 t \exp\Big(C_1 \int_0^s |\nabla u(\cdot,s)|_0\, ds\Big) \tag{3.38}$$

$$\cdot \int_0^t \exp\big(\int_0^s |\nabla u(\cdot,\tau)|_0\, d\tau\big)\, ds. \tag{3.39}$$

Thus, if $\int_0^t |\nabla u(\cdot,s)|\, ds < \infty$ for all $t \in [0,T]$, then the gradient term $|\nabla_a \mathcal{X}(t)|_{1,\gamma}$ is a priori bounded, and it follows that $|\mathcal{X}|_{1,\gamma}$ is also bounded.

Step 2:

The next step is to show that for each $t \in [0,T]$ the gradient term $\int_0^t |\nabla u(\cdot,s)|\, ds$ can be controlled by the vorticity term $\int_0^t |\omega(\cdot,s)|_0\, ds$. This proof is highly technical and makes use of the representation formula (3.17) for the gradient of the velocity field u. The final result is that there exists a constant $C = C(\omega_0)$ such that for all $t \in [0,T]$ it holds that

$$1 + \int_0^t |\nabla u(\cdot,s)|_0\, ds \leq \exp\Big(C(\omega_0) \int_0^t |\omega(\cdot,s)|_0\, ds\Big).$$

This last inequality combined with Inequality (3.38) completes the proof.□

An immediate consequence of the Beale-Kato-Majda theorem 3.14 is the global existence of classical solutions to the incompressible Euler equations (3.29) in two spatial dimensions.

Theorem 3.16 *Let* $\gamma \in (0,1)$*, let the Banach space* $\mathbb{B}$ *be defined by* (3.31) *and consider the vorticity-stream function formulation of the incompressible Euler equations in two-spatial dimensions* (3.5)*, with compactly supported initial vorticity* ω_0 *such that* $\|\omega_0\|_\gamma < \infty$*. Then there exists a unique global solution* $\mathcal{X}(\cdot,t) \in C^1\big([0,\infty);\mathbb{B}\big)$ *to the corresponding Lagrangian formulation of the incompressible Euler equation* (3.29) *for all time* $t \in [0,\infty)$*.*

Proof Let $T > 0$. We recall that in the case of two spatial dimensions, the vorticity ω is preserved along the Lagrangian trajectories, i.e., for all $a \in \mathbb{R}^2, t \in [0,T]$ it holds that

$$\omega\big(\mathcal{X}(a,t),t\big) = \omega_0(a).$$

Hence, for all $t \in [0,T]$ it holds that

$$|\omega(\cdot,t)|_0 \equiv |\omega_0|_0,$$

and thus

$$\int_0^T |\omega(\cdot,s)|_0 \, ds \leq |\omega_0|_0 T.$$

We can therefore apply the Beale-Kato-Majda theorem 3.14 in order to obtain global existence of a unique classical solution. □

Remark 3.17 *Note that the hypothesis of the Beale-Kato-Majda theorem 3.14 requires some regularity constraint on the initial vorticity. However, many two-dimensional flows involve initial vorticities with significantly less regularity, such as so-called* vortex patches $\big(\omega_0 \in L^\infty(\mathbb{R}^2)\big)$ *or so-called* shear layers *(the initial vorticity is a measure). In such cases, we cannot apply Theorem 3.14, and we therefore consider these flows in the next chapter.*

3.3 Weak Solutions of the 2-D Incompressible Euler Equations

We have thus far discussed three equivalent formulations of the incompressible Euler equations in two spatial dimensions:

1. **Velocity-pressure formulation** (3.2):

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= 0 && \text{in } \mathbb{R}^2 \times (0,T], \\ \operatorname{div} u &= 0 && \text{in } \mathbb{R}^2 \times (0,T], \\ u(x,0) &= u_0(x) && \text{in } \mathbb{R}^2; \end{aligned} \tag{3.40}$$

2. **Vorticity-stream function formulation** (3.5):

$$
\begin{aligned}
\frac{\mathrm{D}\omega}{\mathrm{D}t} := \omega_t + (u \cdot \nabla)\omega = 0 \qquad & \text{in } \mathbb{R}^2 \times (0,T], \\
\omega(x,0) = \omega_0(x) \qquad & \text{in } \mathbb{R}^2, \\
u(x,t) = \int_{\mathbb{R}^2} K(x-y)\omega(y,t), & \\
K(x) = \frac{1}{2\pi}\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \Big); &
\end{aligned}
\tag{3.41}
$$

3. **Lagrangian formulation** (3.29):

$$
\begin{aligned}
\frac{d\mathcal{X}(a,t)}{dt} &= u\big(\mathcal{X}(a,t),t\big), \\
\mathcal{X}(a,0) &= a, \\
u\big(\mathcal{X}(a,t),t\big) &= \int_{\mathbb{R}^2} K\big(\mathcal{X}(a,t) - \mathcal{X}(\bar{a},t)\omega_0(\bar{a})\, d\bar{a}\big), \\
K(x) &= \frac{1}{2\pi}\Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \Big);
\end{aligned}
\tag{3.42}
$$

In Section 3.2, we studied the well-posedness of the classical solutions to the incompressible Euler equations (3.41) and (3.42) under the assumption that the initial data ω_0 was sufficiently smooth. We are now interested in problems with weaker regularity constraints on the initial data ω_0, such as, for example, vortex patches corresponding to

$$\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2).$$

As is usual in such cases, we will consider *weak solutions* to the incompressible Euler equations (3.41). To this end, consider the following calculation: let $T > 0$ and let $\phi \in C^1\big(\mathbb{R}^2 \times [0,T]\big)$ be a test function compactly supported in space. Then it holds that

$$\frac{d}{dt}\int_{\mathbb{R}^2} \phi(x,t)\omega(x,t)\,dx = \int_{\mathbb{R}^2} \frac{\mathrm{D}}{\mathrm{D}t}\big(\phi\omega\big)\,dx \ = \int_{\mathbb{R}^2} \omega\frac{\mathrm{D}\phi}{\mathrm{D}t}\,dx + \int_{\mathbb{R}^2} \phi\frac{\mathrm{D}\omega}{\mathrm{D}t}\,dx.$$

Note that if ω solves the initial value problem (3.41) in the classical sense, then the right-hand side of the above equation is identically equal to zero.

Integrating the above equation on the interval $[0,T]$, we obtain

$$\int_{\mathbb{R}^2} \phi(x,T)\omega(x,T)\,dx - \int_{\mathbb{R}^2} \phi(x,0)\omega(x,0)\,dx \\ = \int_0^T \int_{\mathbb{R}^2} \omega\big(\phi_t + u(\cdot\nabla)\phi\big)\,dxdt. \tag{3.43}$$

Equation (3.43) therefore suggests the following definition of a weak solution to the incompressible Euler equations (3.41):

Definition 3.18 *Let $T>0$, let $\omega_0 \in L^1(\mathbb{R}^2)\cap\mathbb{L}^\infty(\mathbb{R}^2)$, let $\omega\colon \mathbb{R}^2\times[0,T]\to\mathbb{R}$ and $u\colon \mathbb{R}^2\times[0,T]\to\mathbb{R}^2$ be functions with the property that*

1. $\omega \in L^\infty\big([0,T]; L^1(\mathbb{R}^2)\cap L^2\infty(\mathbb{R}^2)\big)$*;*
2. $u = k*\omega, \quad \omega = \operatorname{curl} u$*;*
3. *for all test functions $\phi \in C^1\big([0,T]; C_0^1(\mathbb{R}^2)\big)$ it holds that*

$$\int_{\mathbb{R}^2} \phi(x,T)\omega(x,T)\,dx - \int_{\mathbb{R}^2} \phi(x,0)\omega(x,0)\,dx \\ = \int_0^T \int_{\mathbb{R}^2} \omega\big(\phi_t + u(\cdot\nabla)\phi\big)\,dxdt.$$

Then, we say that (ω,u) is a weak solution to the vorticity-stream function formulation of the incompressible Euler equations (3.41) *with initial datum ω_0.*

Exercise 3.19 *Show that*

1. *every smooth solution ω of the IVP* (3.41) *is also a weak solution of the IVP* (3.41)*;*
2. *if (ω,u) is a C^1-weak solutions, then both ω and u are smooth functions.*

We now have the following existence result for weak solutions to the IVP (3.41):

Theorem 3.20 *Let $\omega_0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Then there exists a weak solution (ω,u) to the vorticity-stream function formulation of the incompressible Euler equations* (3.41) *in the sense of Definition 3.18 for all time $t\in[0,\infty)$.*

Proof The proof of Theorem 3.20 is based on the use of mollifiers. We will proceed in three steps.

1. We will construct a sequence of approximate solutions with the desired properties;
2. we will extract a convergent subsequence from this sequence of functions;
3. we will show that the limit of the subsequence is a weak solution to IVP (3.41).

Step 1:

Let $\epsilon > 0$ and let $\rho \in C_0^\infty(\mathbb{R}^2)$ be a non-negative smooth function with the property that

$$\int_{\mathbb{R}^2} \rho \, dx = 1.$$

We denote by ω_0^ϵ the mollification of the initial data ω_0 with ρ_ϵ:

$$\omega_0^\epsilon := \rho_\epsilon * \omega_0 = \frac{1}{\epsilon^2} \int_{\mathbb{R}^2} \rho_\epsilon\big(\frac{x-y}{\epsilon}\big)\omega_0(y)\, dy.$$

It is then straightforward to check that

$$\begin{aligned} \|\omega_0^\epsilon\|_{L^\infty} &\leq \|\omega_0\|_{L^\infty}, \\ \|\omega_0^\epsilon\|_{L^1} &\leq \|\omega_0\|_{L^1}, \\ \lim_{\epsilon\to 0} \|\omega_0^\epsilon - \omega_0\|_{L^1} &= 0. \end{aligned} \tag{3.44}$$

Next, observe that for any given $\epsilon > 0$, the function $\omega_0^\epsilon \in C_0^\infty(\mathbb{R}^2)$ and thus, Theorem 3.16 implies the existence of a global smooth solution $(u^\epsilon, \omega^\epsilon)$ for all times $T > 0$ such that $u^\epsilon = K * \omega^\epsilon$ with K defined by Equation (3.41).

It follows that $(u^\epsilon, \omega^\epsilon)$ is also a weak solution to IVP (3.41) and for all test functions $\phi \in C^1\big([0,T]; C_0^1(\mathbb{R}^2)\big)$ it holds that

$$\int_{\mathbb{R}^2} \phi(x,T)\omega^\epsilon(x,T)\, dx - \int_{\mathbb{R}^2} \phi(x,0)\omega_0^\epsilon(x)\, dx \tag{3.45}$$

$$= \int_0^T \int_{\mathbb{R}^2} \omega^\epsilon\big(\phi_t + u^\epsilon(\cdot\nabla)\phi\big)\, dxdt. \tag{3.46}$$

Next, we prove some useful estimates on the approximate solutions ω^ϵ and u^ϵ. We introduce the mixed norm $|||\cdot|||\colon L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)\to\mathbb{R}_+$ as the function with the property that for all $g\in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ it holds that

$$|||g||| = \|g\|_{L^1} + \|g\|_{L^\infty}.$$

It can then be shown that there exists some constant $C>0$ such that for all $t\in[0,\infty)$ it holds that

$$\|u^\epsilon(\cdot,t)\|_{L^\infty} \le C|||\omega^\epsilon(\cdot,t)||| \le C|||\omega_0|||. \tag{3.47}$$

Indeed, observe that the regularised vorticity ω^ϵ is preserved along the Lagrangian trajectories $\mathcal{X}\colon\mathbb{R}^2\times\mathbb{R}_+\to\mathbb{R}^2$:

$$\omega^\epsilon\big(\mathcal{X}(a,t),t\big) = \omega_0^\epsilon(a) \quad \forall\, a\in\mathbb{R}^2,$$

and it is therefore an easy exercise to show that there exists some constant $C_1>0$ such that for all $t\in[0,\infty)$ it holds that

$$|||\omega^\epsilon(\cdot,t)||| \le C_1|||\omega_0^\epsilon||| \le C_1|||\omega_0|||.$$

Next, note that by definition the regularised velocity field is given by

$$u^\epsilon(x,t) = \int_{\mathbb{R}^2} K(x-y)\omega^\epsilon(y,t)\,dy.$$

Let $\bar\rho\in C_0^\infty(\mathbb{R}^2)$ be a cutoff function defined by

$$\bar\rho(x) = \begin{cases} 1 & \text{if } |x|\le 1,\\ 0 & \text{if} |x|\ge 2.\end{cases} \tag{3.48}$$

It follows that

$$u^\epsilon(x,t) = \underbrace{\int_{\mathbb{R}^2}\bar\rho(x-y)K(x-y)\omega^\epsilon(y,t)\,dy}_{:=u_1^\epsilon} + \underbrace{\int_{\mathbb{R}^2}(1-\bar\rho(x-y))K(x-y)\omega^\epsilon(y,t)\,dy}_{:=u_2^\epsilon}.$$

Using the fact that there exists a constant $C_2>0$ such that for all $x\in\mathbb{R}^2, x\ne 0$ it holds that

$$|K(x)| \le \frac{C_2}{|x|},$$

together with Young's inequality, we obtain the existence of a constant $K > 0$ such that for all $t \in [0, \infty)$ it holds that

$$\begin{aligned} \|u^\epsilon(\cdot,t)\|_{L^\infty} \leq & \quad \|\omega^\epsilon(\cdot,t)\|_{L^\infty}\|\bar{\rho}K\|_{L^1} + \|(1-\bar{\rho})K\|_{L^\infty}\|\omega^\epsilon(\cdot,t)\|_{L^1} \\ & \leq K\||u(\cdot,t)\||, \end{aligned}$$

thus proving Inequality (3.47).

Step 2:

The next step is to extract suitable subsequences $(u^{\epsilon'}), (\omega^{\epsilon'})$ from the approximate solutions (u^ϵ) and (ω^ϵ) respectively. We therefore introduce some notation.

For a given $t \in [0, T]$, we denote by $\mathcal{X}^{-t}\colon \mathbb{R}^2 \to \mathbb{R}^2$ the function with the property that for all $a, x \in \mathbb{R}^2$ it holds that

$$\mathcal{X}\big(\mathcal{X}^{-t}(x), t\big) = x, \qquad \text{and} \mathcal{X}^{-t}\big(\mathcal{X}(a,t)\big) = a.$$

Thus, for each $t \in [0, T]$, $\mathcal{X}^(- t)$ is simply the reverse flow-map. Next, we denote by $\mathcal{X}^\epsilon$ the flow-map generated by the approximate solution u^ϵ and we denote by $\mathcal{X}_\epsilon^{-t}$ the reverse flow-map at time $t \in [0, T]$ associated with the approximate solution u^ϵ.

Using this notation, we observe that the approximate solutions u^ϵ, ω^ϵ satisfy the following Lagrangian formulation of the incompressible Euler equations:

$$\begin{aligned} \frac{d\mathcal{X}^\epsilon}{dt} &= u^\epsilon\big(\mathcal{X}^\epsilon(a,t), t\big), \\ \mathcal{X}^\epsilon(a, 0) &= a, \\ \omega^\epsilon(x,t) &= \omega_0^\epsilon\big(\mathcal{X}_\epsilon^{-t}(x)\big). \end{aligned} \tag{3.49}$$

Thus, instead of considering subsequences $(u^{\epsilon'}), (\omega^{\epsilon'})$, we can construct a suitable subsequence $\mathcal{X}^{\epsilon'}$ and pass to the limit in the *Lagrangian trajectories*:

$$\mathcal{X}^{\epsilon'} \to \mathcal{X} \qquad \text{locally uniformly in x,}$$

We can then define the weak solution (ω, u) to the IVP (3.41) as

$$\begin{aligned} \omega(x,t) &= \omega_0\big(\mathcal{X}^{-t}(x)\big), \\ u(x,t) &= K * \omega. \end{aligned}$$

In order to show the existence of the convergent subsequence $(\mathcal{X}^\epsilon)$, we will make use of the *Arzela-Ascoli* theorem. Therefore, we first prove some uniform (in ϵ) bounds on the trajectories $(\mathcal{X}^\epsilon)$. We require the following lemma from Potential theory:

Lemma 3.21 *Let $T > 0$, let $\omega_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and let ω^ϵ and u^ϵ be the smooth solutions on the time interval $[0, T]$ to the vorticity-stream function formulation of the incompressible Euler equations* (3.41) *with mollified initial data ω_0^ϵ. Then there exists some constant $C = C(T)$ and the exponent $\beta(t) = \exp\big(-C|||\omega_0|||t\big)$ such that for all $\epsilon > 0$ and for all $t \in [0, T]$ it holds that*

$$|\mathcal{X} - \mathcal{X}| \le C|x_1 - x_2|^{\beta(t)}, |\mathcal{X} - \mathcal{X}| \le C|x_1 - x_2|^{\beta(t)},$$

and furthermore for all $\epsilon > 0$ and for all $t_1, t_2 \in [0, T]$ it holds that

$$|\mathcal{X} - \mathcal{X}| \le C|x_1 - x_2|^{\beta(t)} |\mathcal{X} - \mathcal{X}| \le C|x_1 - x_2|^{\beta(t)}$$

and finally for all $t \in [0, T]$ it holds that $u^\epsilon(\cdot, t)$ is quasi-Lipschitz continuous, i.e., there exists some constant $L > 0$ such that for all $x_1, x_2 \in \mathbb{R}^2$ it holds that

$$\sup_{0 \le t \le T} |u^\epsilon(x_1, t) - |^\epsilon(x_2, t)| \le L|||\omega_0||||x_1 - x_2|\big(1 - \log^-(|x_1 - x_2|)\big), \quad (3.50)$$

where $\log^-(a) = \log(a)$ *for* $a \in (0, 1)$ *and* $\log^-(a) = 0$ *for* $a \ge 1$.

Proof Lemma 3.21 and its proof can be found in [MB02, Lemma 8.1-8.2].□

Now, we observe that by definition, for all $x \in \mathbb{R}^2$ it holds that

$$\begin{aligned} |\mathcal{X}_\epsilon^{-t} - x| = |\mathcal{X}_\epsilon(a, t) - a| &= \left| \int_0^t u^\epsilon\big(\mathcal{X}_\epsilon(a, s), s\big)\, ds \right| \\ &\le \bar{C}T, \end{aligned}$$

where the last inequality follows from the uniform bound (3.47).

It follows that for all $t \in [0, T]$, the family of reverse flow maps $(\mathcal{X}_\epsilon^{-t})$ is uniformly bounded. Furthermore, Lemma 3.21 can be used to show that the family $(\mathcal{X}_\epsilon^{-t})$ is also equicontinuous on the set $\{x \colon |x| \le R\} \times [0, T]$, where $R > 0$ is a constant.

Similarly, the family of Lagrangian trajectories $\mathcal{X}^\epsilon$ can also be shown to be equicontinuous on the set $\{a\colon |x| \le R\} \times [0,T]$. Thus, we can apply the Arzela-Ascoli theorem to obtain the existence of a subsequence $(\mathcal{X}_{\epsilon'}^{-t})$ such that

$$\mathcal{X}_{\epsilon'}^{-t}(x) \to \mathcal{X}^{-t}(x) \qquad \text{uniformly on } \{x\colon |x| \le R\} \times [0,T].$$

We can now define the solution vorticity ω as $\omega(x,t) = \omega_0\big(\mathcal{X}^{-t}(x)\big)$ and the solution velocity u as $u = K * \omega$, with K given by (3.41).

Next, we claim that for all $t \in [0,T]$ the reverse flow map $\mathcal{X}^{-1}(t)$ is measure-preserving from $\mathbb{R}^2$ to $\mathbb{R}^2$, and for all $t \in [0,T]$ and all $f \in L^1(\mathbb{R}^2)$ it holds that

$$\int_{\mathbb{R}^2} f\big(\mathcal{X}^{-t}(x)\big)\,dx = \int_{\mathbb{R}^2} f(x)\,dx.$$

Proof (Sketch) First, let $f \in C_c^0(\mathbb{R}^2)$. Then the dominated convergence theorem implies that for all $t \in [0,T]$ it holds that

$$\int_{\mathbb{R}^2} f\big(\mathcal{X}^{-t}(x)\big)\,dx = \lim_{\epsilon'\to 0} \int_{\mathbb{R}^2} f\big(\mathcal{X}_{\epsilon'}^{-t}(x)\big)\,dx = \int_{\mathbb{R}^2} f(x)\,dx. \tag{3.51}$$

Using the fact that $C_c^1(\mathbb{R}^2)$ is dense in $L^1(\mathbb{R}^2)$ completes the proof for the second part of the claim. Furthermore, the Reisz representation theorem for measures can be used to prove that for all $t \in [0,T]$ the reverse flow map $\mathcal{X}^{-1}(t)$ is also measure-preserving from $\mathbb{R}^2$ to $\mathbb{R}^2$. □

We will now use the above claim to show that for all $t \in [0,T]$ it holds that $\omega^{\epsilon'}(\cdot,t) \to \omega(\cdot,t)$ in L^1 and $u^{\epsilon'}(\cdot,t) \to u(\cdot,t)$ locally uniformly in space.

Observe that for all $t \in [0,T]$ it holds that

$$\begin{aligned}\lim_{\epsilon'\to 0} \|\omega^{\epsilon'}(\cdot,t) - \omega(\cdot,t)\|_{L^1} &= \lim_{\epsilon'\to 0} \|\omega_0^{\epsilon'}(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0(\mathcal{X}^{-t})\|_{L^1} \\ &\le \underbrace{\lim_{\epsilon'\to 0} \|\omega_0^{\epsilon'}(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0(\mathcal{X}_{\epsilon'}^{-t})\|_{L^1}}_{=T_1} \\ &\quad + \underbrace{\lim_{\epsilon'\to 0} \|\omega_0(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0(\mathcal{X}^{-t})\|_{L^1}}_{=T_2}.\end{aligned}$$

The term T_1 can be simplified using Equation (3.51):

$$\begin{aligned} T_1 &= \lim_{\epsilon' \to 0} \int_{\mathbb{R}^2} |\omega_0^{\epsilon'}(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0(\mathcal{X}_{\epsilon'}^{-t})|\, dx = \lim_{\epsilon' \to 0} \int_{\mathbb{R}^2} |\omega_0^{\epsilon'}(x) - \omega_0(x)|\, dx \\ &= 0. \end{aligned}$$

Furthermore, since $\omega_0 \in L^1(\mathbb{R}^2)$ by hypothesis, there exists a sequence of continuous functions $\{\omega_0^n\}_{n \in \mathbb{N}} \colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$|\omega_0 - \omega_0^n|_{L^1} \leq \frac{1}{n}.$$

Thus, we can also simplify the term T_2 as follows:

$$\begin{aligned} T_2 = \lim_{\epsilon' \to 0} \|\omega_0(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0(\mathcal{X}^{-t})\|_{L^1} \leq & \lim_{\epsilon' \to 0} \|\omega_0(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0^n(\mathcal{X}_{\epsilon'}^{-t})\|_{L^1} \\ &+ \lim_{\epsilon' \to 0} \|\omega_0^n(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0^n(\mathcal{X}^{-t})\|_{L^1} \\ &+ \lim_{\epsilon' \to 0} \|\omega_0^n(\mathcal{X}^{-t}) - \omega_0(\mathcal{X}^{-t})\|_{L^1}. \end{aligned}$$

Note that Equation (3.51) clearly implies that

$$\begin{aligned} \lim_{\epsilon' \to 0} \|\omega_0(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0^n(\mathcal{X}_{\epsilon'}^{-t})\|_{L^1} &= \lim_{\epsilon' \to 0} \|\omega_0 - \omega_0^n\|_{L^1} \leq \frac{1}{n}, \\ \lim_{\epsilon' \to 0} \|\omega_0^n(\mathcal{X}^{-t}) - \omega_0(\mathcal{X}^{-t})\|_{L^1} &= \lim_{\epsilon' \to 0} \|\omega_0^n - \omega_0\|_{L^1} \leq \frac{1}{n}. \end{aligned}$$

Furthermore, the function ω_0^n is continuous by hypothesis and therefore for all $t \in [0, T]$ it holds that $\omega_0^n(\mathcal{X}_{\epsilon'}^{-t}) \to \omega_0^n(\mathcal{X}^{-t})$ point-wise. Thus, the dominated convergence theorem implies that for all $t \in [0, T]$ it holds that

$$\lim_{\epsilon' \to 0} \|\omega_0^n(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0^n(\mathcal{X}^{-t})\|_{L^1} = 0.$$

Hence, we conclude that taking the limit $n \to \infty$, we obtain

$$T_2 = 0.$$

Therefore, we have shown that $\lim_{\epsilon' \to 0} \|\omega^{\epsilon'}(\cdot, t) - \omega(\cdot, t)\|_{L^1} = T_1 + T_2 = 0$.

We next show that $u^{\epsilon'}(\cdot, t) \to u(\cdot, t)$ locally uniformly in space. To this end, let $\rho \in C_0^\infty(\mathbb{R}^2)$ be a cutoff function defined by Equation (3.48), let $\delta > 0$ and let $\rho_\delta \in C_0^\infty(\mathbb{R}^2)$ be the function with the property that for all $x \in \mathbb{R}^2$

$$\rho_\delta(x) = \rho(x/\delta).$$

By definition of the solution velocity u and the approximate velocities $(u)^{\epsilon'}$, for all $t \in [0,T]$ it holds that

$$\begin{aligned} c|u^{\epsilon'}(x,t) - u(x,t)| &= \left| K * \Big(\omega_0^{\epsilon'}\big(\mathcal{X}_{\epsilon'}^{-t}(x)\big) - \omega_0\big(\mathcal{X}^{-t}(x)\big)\Big)\right| \\ &\le \underbrace{\left|\rho_\delta K * \Big(\omega_0^{\epsilon'}\big(\mathcal{X}_{\epsilon'}^{-t}(x)\big) - \omega_0\big(\mathcal{X}^{-t}(x)\big)\Big)\right|}_{=I_1} \\ &\quad + \underbrace{\left|\big((1-\rho_\delta)K\big) * \Big(\omega_0^{\epsilon'}\big(\mathcal{X}_{\epsilon'}^{-t}(x)\big) - \omega_0\big(\mathcal{X}^{-t}(x)\big)\Big)\right|}_{=I_2} \end{aligned}$$

It then follows that there exists some constant $C > 0$ such that

$$\begin{aligned} I_1 &= \left|\rho_\delta K * \Big(\omega_0^{\epsilon'}\big(\mathcal{X}_{\epsilon'}^{-t}(x)\big) - \omega_0\big(\mathcal{X}^{-t}(x)\big)\Big)\right| \\ &\le \|\rho_\delta K\|_{L^1}\Big(\|\omega_0^{\epsilon}(\mathcal{X}_{\epsilon'}^{-t})\|_{L^\infty} + \|\omega_0(\mathcal{X}^{-t})\|_{L^\infty}\Big) \\ &\le 2\|\omega\|_{L^\infty}\int_{|x|\le 2\delta} |\rho_\delta(x)K(x)|\,dx \\ &\le C\int_{|x|\le 2\delta} |x|^{-1}\,dx \le C\delta \end{aligned}$$

Similarly, it also follows that

$$\begin{aligned} I_2 &= \left|\big((1-\rho_\delta)K\big) * \Big(\omega_0^{\epsilon'}\big(\mathcal{X}_{\epsilon'}^{-t}(x)\big) - \omega_0\big(\mathcal{X}^{-t}(x)\big)\Big)\right| \\ &\le \|(1-\rho_\delta)K\|_{L^\infty}\|\omega_0^{\epsilon}(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0(\mathcal{X}^{-t})\|_{L^1} \\ &\le \frac{C}{\delta}\|\omega_0^{\epsilon}(\mathcal{X}_{\epsilon'}^{-t}) - \omega_0(\mathcal{X}^{-t})\|_{L^1}. \end{aligned}$$

Equation (3.51) therefore implies that for all $t \in [0,T]$ it holds that

$$|u^{\epsilon'}(x,t) - u(x,t)| \le C\delta + \frac{C}{\delta}\|\omega^{\epsilon'}(\cdot,t) - \omega(\cdot,t)\|_{L^1}.$$

Next, given any $\eta > 0$, we pick $\delta = \frac{\eta}{2C}$. We have previously shown that

$$\lim_{\epsilon'\to 0}\|\omega^{\epsilon'}(\cdot,t) - \omega(\cdot,t)\|_{L^1} = 0.$$

Hence, we can pick some $\epsilon_0 > 0$ such that for all $\epsilon' \in (0,\epsilon_0]$ it holds that

$$\|\omega^{\epsilon'}(\cdot,t) - \omega(\cdot,t)\|_{L^1} \le \delta^2.$$

Hence, for any $\eta > 0$ there exists $\epsilon_0 > 0$ such that for all $\epsilon' \in (0, \epsilon_0]$, it holds

$$|u^{\epsilon'}(\cdot, t) - u(\cdot, t)| \leq \eta,$$

and therefore $u^{\epsilon'}(\cdot, t) \to u(\cdot, t)$ locally uniformly in space.

Step 3:

To recap, we have shown that (ω, u) is the limit of a suitable subsequence of approximate solution $(\omega^{\epsilon'}, u^{\epsilon'})$ and the limit is obtained in the L^1-sense for the vorticity and locally uniformly in space for the velocity. This classification of the convergence allows us to conclude convergence of the non-linear term $\omega^{\epsilon'} u^{\epsilon'}$ as we will show below.

In order to complete the proof, we must demonstrate that (ω, u) is indeed a solution to the vorticity-stream function formulation of the incompressible Euler equations (3.41).

Since, $(\omega^{\epsilon'}, u^{\epsilon'})$ is a weak solution to IVP (3.41) with mollified initial data $\omega_0^{\epsilon'}$, for all test functions $\phi \in C^1\big([0,T]; C_0^1(\mathbb{R}^2)\big)$ it holds that

$$\begin{aligned}\int_{\mathbb{R}^2} \phi(x,T)\omega^{\epsilon'}(x,T)\,dx - \int_{\mathbb{R}^2} \phi(x,0)\omega_0^{\epsilon'}(x)\,dx \\ = \int_0^T \int_{\mathbb{R}^2} \omega^{\epsilon'}\big(\phi_t + u^{\epsilon'}(\cdot\nabla)\phi\big)\,dxdt.\end{aligned}$$

Taking the limit $\epsilon' \to 0$, we obtain

$$\begin{aligned}\underbrace{\lim_{\epsilon'\to 0}\int_{\mathbb{R}^2} \phi(x,T)\omega^{\epsilon'}(x,T)\,dx}_{=T_1} - \underbrace{\lim_{\epsilon'\to 0}\int_{\mathbb{R}^2} \phi(x,0)\omega_0^{\epsilon'}(x)\,dx}_{=T_2} \\ = \underbrace{\lim_{\epsilon'\to 0}\int_0^T \int_{\mathbb{R}^2} \omega^{\epsilon'}\big(\phi_t + u^{\epsilon'}(\cdot\nabla)\phi\big)\,dxdt}_{=T_3}.\end{aligned}$$

Since for all $t \in [0,T]$, $\omega^{\epsilon'}(\cdot, t) \to \omega(\cdot, t)$ in the L^1-sense, it follows that

$$T_1 = \lim_{\epsilon'\to 0}\int_{\mathbb{R}^2} \phi(x,T)\omega^{\epsilon'}(x,T)\,dx = \int_{\mathbb{R}^2} \phi(x,T)\omega(x,T)\,dx,$$

and similarly

$$T_2 = \lim_{\epsilon'\to 0}\int_{\mathbb{R}^2} \phi(x,0)\omega_0^{\epsilon'}(x)\,dx = \int_{\mathbb{R}^2} \phi(x,0)\omega_0(x).$$

Finally, the term T_3 can be written as

$$
\begin{aligned}
T_3 &= \lim_{\epsilon'\to 0}\int_0^T\int_{\mathbb{R}^2}\omega^{\epsilon'}\big(\phi_t + u^{\epsilon'}(\cdot\nabla)\phi\big)\,dxdt \\
&= \underbrace{\lim_{\epsilon'\to 0}\int_0^T\int_{\mathbb{R}^2}\omega^{\epsilon'}\phi_t\,dxdt}_{=I_1} + \underbrace{\lim_{\epsilon'\to 0}\int_0^T\int_{\mathbb{R}^2}\omega^{\epsilon'}\big(u^{\epsilon'}(\cdot\nabla)\phi\big)\,dxdt.}_{=I_2}
\end{aligned}
$$

Again, using the fact that for all $t\in[0,T]$, $\omega^{\epsilon'}(\cdot,t)\to\omega(\cdot,t)$ in the L^1-sense, the term I_1 reduces to

$$
\lim_{\epsilon'\to 0}\int_0^T\int_{\mathbb{R}^2}\omega^{\epsilon'}\phi_t\,dxdt = \int_0^T\int_{\mathbb{R}^2}\omega\phi_t\,dxdt,
$$

and using the fact that for all $t\in[0,T]$ it holds that $\omega^{\epsilon'}(\cdot,t)\to\omega(\cdot,t)$ in the L^1-sense and $u^{\epsilon'}(\cdot,t)\to u(\cdot,t)$ locally uniformly in space we obtain

$$
I_2 = \lim_{\epsilon'\to 0}\int_0^T\int_{\mathbb{R}^2}\omega^{\epsilon'}\big(u^{\epsilon'}(\cdot\nabla)\phi\big)\,dxdt = \int_0^T\int_{\mathbb{R}^2}\omega\big(u(\cdot\nabla)\phi\big)\,dxdt.
$$

Combining these limits together, we obtain that (ω,u) satisfies Equation (3.43), and therefore (ω,u) is indeed a weak solution of the vorticity-stream function formulation of the incompressible Euler equations (3.41). The proof is thus complete. □

Remark 3.22 *For more details on Theorem 3.20 as well as proofs of the potential theory estimates we have used, the reader can consult [MB02, Theorem 8.1].*

We conclude this section by presenting a final theorem on the uniqueness of weak solutions to the vorticity-stream function formulation of the incompressible Euler equations (3.41).

Theorem 3.23 *Let $\omega_0\in L_0^\infty(\mathbb{R}^2)$ be a function with the property that there exists some constant $R_0>0$ such that*

$$
supp\,\omega_0 \subset \{x\colon |x|\le R_0\}.
$$

Then the weak solution (ω,u), $\omega\in L^\infty\big([0,\infty);L_0^\infty(\mathbb{R}^2)\big)$ of the vorticity-stream function formulation of the incompressible Euler equations (3.41) *with initial datum ω_0 is unique.*

Proof We follow the proof presented by V. I. Yudovich in 1963 [Yud63]. We will use the following lemma, which is based on the Calderon-Zygmund potential theory estimate:

Lemma 3.24 *Let $T > 0$, let $R\colon [0,T] \to \mathbb{R}$ and let $\omega(\cdot,t) \in L_0^\infty(\mathbb{R}^2)$ be a weak solution at time $t \in [0,T]$ to the vorticity-stream function formulation of the incompressible Euler equations* (3.41) *such that supp$\omega(\cdot,t) \subset \{x\colon |x| \le R(t)\}$. Then there exists a constant $\bar{C} > 0$ such that for all $p \in (1,\infty)$ it holds that*

$$\|\nabla u(\cdot,t)\|_{L^p} \le \bar{C}(\|\omega_0\|_{L^\infty})p. \tag{3.52}$$

Proof Lemma 3.24 and its proof can be found in [MB02, Lemma 8.3]. □

We begin the proof by showing first that any weak solution ω satisfying the hypothesis of Theorem 3.23 also satisfies for all $t \in [0,T]$:

$$\int_{\mathbb{R}^2} \omega(x,t)\,dx = \int_{\mathbb{R}^2} \omega_0(x)\,dx.$$

To this end, note that any solution $\omega(\cdot,t) \in L^1 \cap L^\infty$ has uniformly bounded velocity $u(\cdot,t) = K * \omega(\cdot,t)$. Indeed, there exists some constant $C > 0$ such that for all $t \in [0,T]$ it holds that

$$\|u(\cdot,t)\|_{L^\infty} \le C\||\omega_0|\|. \tag{3.53}$$

Furthermore, since the initial vorticity flows under the action of the Lagrangian trajectories, which are volume preserving, it follows that there exists an increasing, bounded function $R\colon [0,T] \to \mathbb{R}$ such that for all $t \in [0,T]$ it holds that

$$\text{supp}\omega(\cdot,t) \subset \{x\colon |x| \le R(t)\}.$$

Moreover, since ω is a weak solution, by definition for all test function $\phi \in C^1\big([0,T]; C_)^1(\mathbb{R}^2)\big)$ it holds that

$$\int_{\mathbb{R}^2} \phi(x,T)\omega(x,T)\,dx - \int_{\mathbb{R}^2} \phi(x,0)\omega(x,0)\,dx = \int_0^T \int_{\mathbb{R}^2} \omega\big(\phi_t + u(\cdot\nabla)\phi\big)\,dxdt.$$

In particular, we may pick a test function ϕ such that $\phi(x,t) \equiv 0$ for all $t \in (0,T]$ and all $|x| \le R(T)$. Substituting this test function ϕ in the above equation, we obtain that the right-hand side is zero and thus for all $t \in [0,T]$ it holds that

$$\int_{\mathbb{R}^2} \omega(x,t)\,dx = \int_{\mathbb{R}^2} \omega_0(x)\,dx,$$

and this proves the claim.

Next, let $(\omega_1, u_1), (\omega_2, u_2)$ be two weak solutions of the IVP (3.41) with the same initial datum $\omega_0 \in L_0^\infty(\mathbb{R}^2)$ such that $\mathrm{supp}\omega_0 \subset \{x \colon |x| \le R_0\}$ for some $R_0 > 0$ and with associated pressure fields P_1 and P_2 respectively. Then the velocities $\{u_j\}_{j=1,2}$ satisfy the velocity pressure formulation of the incompressible Euler equations (3.40) in the sense of distributions:

$$\begin{aligned} \frac{\partial}{\partial t} u_j + (u_j \cdot \nabla) u_j &= -\nabla p_j, \\ \nabla \cdot u_j &= 0. \end{aligned}$$

We now define $u^* = u_1 - u_2$. Since for all $t \in [0,T]$ and $j = 1,2$ it holds that $\mathrm{supp}\omega_j(\cdot,t) \subset \{x \colon |x| \le R_j(T)\}$, we can use an asymptotic expansion of the kernel K to obtain

$$u_j(x,t) = \frac{\tilde{C}}{|x|} \int_{\mathbb{R}^2} \omega_j(y,t)\,dy + \mathcal{O}\big(|x|^{-2}\big) \qquad \text{for } |x| \ge 2R_j(T),$$

where $\tilde{C} > 0$ is some constant.

It therefore follows that

$$\begin{aligned} u^*(x,t) &= \frac{C}{|x|}\left(\int_{\mathbb{R}^2} \omega_1(y,t)\,dy - \int_{\mathbb{R}^2} \omega_1(y,t)\,dy\right) + \mathcal{O}\big(|x|^{-2}\big) \\ \underbrace{\Longrightarrow}_{\substack{\text{Eq.}\\ (3.53)}} u^*(x,t) &= \frac{C}{|x|}\left(\int_{\mathbb{R}^2} \omega_0(y)\,dy - \int_{\mathbb{R}^2} \omega_0(y)\,dy\right) + \mathcal{O}\big(|x|^{-2}\big) \\ &\sim \mathcal{O}\big(|x|^{-2}\big). \end{aligned}$$

Thus, we obtain that $u^* = u_1 - u_2$ has finite energy E at all time $t \in [0,T]$:

$$E(t) := \int_{\mathbb{R}^2} |u^*(x,t)|^2\,dx < \infty.$$

Next, we observe that the function u^* also satisfies

$$u_t^* + (u_1 \cdot \nabla) u^* + (u^* \cdot \nabla) u_2 = -\nabla(P_1 - P_2),$$

in the sense of distributions. Moreover, taking the inner product with the function u^* of the above equation and integrating by parts yields for all $t \in [0,T]$:

$$\frac{1}{2}\frac{d}{dt}E(t) - \int_{\mathbb{R}^2} |u^*|^2 \operatorname{div} u_1 + \int_{\mathbb{R}^2} (u^* \cdot \nabla) u_2 \cdot u^* \, dx = \int_{\mathbb{R}^2} (P_1 - P_2) \operatorname{div} u^*.$$

Using the fact that the velocities $\{u_j\}_{j=1,2}$ are divergence free together with Hölder's inequality we obtain that for all $p \in (1,\infty)$ it holds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E(t) &\le 2\int_{\mathbb{R}^2} |u^*|^2 |\nabla u_2| \, dx \\ &\le 2\|\nabla u_2\|_{L^p} \left(\int_{\mathbb{R}^2} |u^*|^{\frac{2p}{p-1}}\right)^{1-1/p} \\ &\le 2\|\nabla u_2\|_{L^p} \|\omega(\cdot,t)\|_{L^\infty}^{\frac{2(p-1)}{(p-1)p}} E(t)^{1-1/p}.\end{aligned}$$

Finally, we can apply (3.52) from Lemma 3.24 together with the fact that the velocities $\{u_j\}_{j=1,2}$ are uniformly bounded to obtain that for all $t \in [0,T]$ it holds that

$$\frac{d}{dt}E(t) \le pME(t)^{1-1/p}, \tag{3.54}$$

where $M = \hat{C}(\|\omega_0\|_{L^\infty})\|\omega_0\|_{L^\infty}^{2/p}$, with $\hat{C}$ begin a constant that depends on $\|\omega_0\|_{L^\infty}$.

It can now be checked that the maximal solution to the differential inequality (3.54) is given by

$$\tilde{E}(t) = (Mt)^p,$$

and therefore we must have for all $t \in [0,T]$

$$E(t) \le \tilde{E}(t).$$

Let us consider an interval $[0,T^*]$ such that $MT^* \le \frac{1}{2}$. Then, clearly for all $t \in [0,T^*]$ it holds that

$$\lim_{p\to\infty} E(t) \le \lim_{p\to\infty} \frac{1}{2^p} = 0,$$

so that $E(t) \equiv 0$ on the interval $[0,T^*]$. The final step is to iterate this argument to conclude that $E(t) \equiv 0$ on the interval $[0,T]$. Thus, we must have $u_1 = u_2$ and therefore $\omega_1 = \omega_2$. Hence, the proof is complete. □

4 Numerical methods

In this chapter, we introduce some basic numerical method of the approximation of solutions for the Navier-Stokes equations (NSE) and for the Euler equations (EE).

There are four widely used types of numerical methods for the solution of NSE and Euler equations:

- Finite difference/finite volume methods (FD/FVM) (see section 4.3):
  - based on Velocity/Pressure formulation (cf. [KL66, Cho68, BCG89]);
  - based on Vorticity/Stream-function formulation (cf. [LT97]);
- Finite element methods (FEM) (cf. [Tem01, GR86, Glo03]), which will not be treated in the following lecture notes;
- (Fourier) spectral methods: based on approximations of the Euler/NSE in Fourier space, will e considered in section 4.1 (cf. [GO81, BT15]);
- Particle-trajectory methods (vortex methods) (cf. [MB02]), solving the ODE arising from the discretization based on the Lagrangian formulation of NSE. Those methods will not be threaded in this lecture notes.

We will concentrate our attention to spectral methods (section 4.1) and finite difference methods (section 4.3).

When approximating the NSE, the following difficulties arise:

- The nonlinear term greatly affects stability of the scheme and is generally the main difficulty in analysing the numerical approximations.
- The constraint $\operatorname{div} u = 0$ and the term ∇p generally have a strong impact on performance of the scheme. Treatment of boundaries for this terms is usually difficult. Pressure has no physical role, and generally must be somehow removed.

- Being time dependant, the NSE require suitable time-stepping algorithms, with obvious implications for efficiency and stability.

4.1 (Fourier) spectral methods

In the following, we let $u \in L_t^\infty L_x^2(\mathbb{T}^d)^d$, $d = 2, 3$, where $\mathbb{T}^d := [0, 1)^d$ denotes the d-dimensional torus. We fix a time interval $I := [0, T]$, $T > 0$. The idea is to rewrite the function u using a Fourier expansion, and use the properties of such expansion to approximate the Euler/NSE equations.

4.1.1 Basic Fourier theory

We briefly recall some basic definition and property of Fourier series.

Theorem 4.1 *The functions*

$$\varphi_k(x) := e^{2\pi i k\cdot x} \in L_x^2(\mathbb{T}^d; \mathbb{C}), \forall k \in \mathbb{Z}^d \tag{4.1}$$

define an orthonormal basis *(ONB) for $L_x^2(\mathbb{T}^d; \mathbb{C})$, relative to the standard inner product $\langle g, f\rangle := \int_0^1 f(x)g(x)^* dx$, i.e. $\langle \varphi_{k'}, \varphi_k\rangle = \delta_{k,k'}$ and φ_k span $L_x^2(\mathbb{T}^d; \mathbb{C})$. We will call the basis functions φ_k* the k-th Fourier modes*.*

Proof Exercise. □

Definition 4.2 *If $u \in C([O, T], L^2(\mathbb{T}^d; \mathbb{C})^d)$, we can write the* Fourier expansion

$$u(x, t) = \sum_{k\in\mathbb{Z}^d} \hat{u}_k(t)\varphi_k(x),\ \forall t \in [0, T], x \in \mathbb{T}^d, \tag{4.2}$$

where define the Fourier coefficients *as*

$$\hat{u}_k(t) := \langle \varphi_k, u(\cdot, t)\rangle \in \mathbb{C}^d, \forall k \in \mathbb{Z}^d, t \in [0, T]. \tag{4.3}$$

Remark 4.3 *More in general, spectral methods could be defined for a different ONB φ_k.*

Armed with this knowledge, it is natural to truncate the Fourier expansion to a finite set of modes, and investigate the convergence of the truncated series.

Definition 4.4 *Let $K > 0$. We define the* truncated Fourier expansion *of u as the function defined by*

$$P_K(u(x,t)) = \sum_{\substack{k\in\mathbb{Z}^d \\ |k|<K}} \hat{u}_k(t)\varphi_k(x). \tag{4.4}$$

Remark 4.5 *The convergence of the series as $K \to \infty$ has to be considered carefully.*

1. *For $d > 1$, the choice of the "shape" of the truncation has an impact on the behaviour of the convergence of the series. Convergence results may differ if we choose a different norm in $|k| < K$ (e.g. maximum norm).*
2. *In 1D, if $u \in C^0(\mathbb{T}) \cap BV(\mathbb{T})$, $BV(\mathbb{T}) := \{u \mid V(u) < \infty\}$ is the space of functions with* bounded variation*, then (4.4) converges absolutely. Recall that $V(u) := \sup_{\mathcal{P}} \sum_{[x_i,x_{i+1}]\in\mathcal{P}} |u(x_{i+1}) - u(x_i)|$, where $\mathcal{P}$ is a partition of $\mathbb{T}$.*
3. *If $u \in BV(\mathbb{T})$, then (4.4) converges pointwise to the average of the jump values:*

$$P_K(u)(x) \to \lim_{y\to x^-} \frac{u(y)}{2} + \lim_{y\to x^+} \frac{u(y)}{2} \tag{4.5}$$

   (the limits exists but may differ).

4. *In general, $u \in C^0(\mathbb{T})$ doesn't imply that the Fourier series converges pointwise.*
5. *The most comprehensive notion of convergence for Fourier series is therefore (strong) L^2 convergence which also holds in multi-d:*

$$\|u - P_K(u)\| \to 0. \tag{4.6}$$

   This is the notion of convergence of interest to us.

From now on, we will consider convergence in L^2_x. We have the following theorem:

Theorem 4.6 *Let $u \in L^2(\mathbb{T}^d)$, then the following holds:*

1. Parseval identity: $\|u\|_{L^2} = \|\{\hat{u}_k\}_{k\in\mathbb{Z}^d}\|_{l^2(\mathbb{Z}^d)}$;

2. $\|P_K(u)\|_{L^2} \le \|u\|_{L^2}$ *and* $\|u - P_K(u)\|_{L^2} \to 0$*;*

3. φ_k *are also a orthogonal basis for* $H^s(\mathbb{T}^d)$*, equipped with the standard inner product* $\langle \cdot, \cdot \rangle_{H^s}$*. In particular, also the Parseval equality holds in this case;*

4. $\|u - P_K(u)\|_{H^r} \le CK^{r-s} \|u\|_{H^s}, \quad \forall s > r$*;*

5. $\|u - P_K(u)\|_{L^\infty} \le CK^{\frac{d}{2}-s} \|u\|_{H^s}, \quad \forall s > \frac{d}{2}$.

Proof 1. No proof. Notice

$$0 \le \|u - P_K(u)\|^2 \le \|u\|^2 - \sum_{\substack{k\in\mathbb{Z}^d \\ |k|<K}} |\langle \varphi_k, u\rangle|^2 .$$

This gives *Bessel's inequality*:

$$\|u\|^2 \ge \|\hat{u}\|_{l^2}^2 .$$

2. We have

$$\langle \varphi_k, u - P_K(u)\rangle = \hat{u}_k - \hat{u}_k = 0, \forall\, |k| < K.$$

Hence,

$$0 = \langle P_K(u), u - P_K(u)\rangle = \frac{1}{2}\|u\|^2 - \frac{1}{2}\|P_K(u)\|^2 - \frac{1}{2}\|u - P_K(u)\|^2 .$$

This implies:

$$\|P_K(u)\|^2 = \|u\|^2 - \|u - P_K(u)\|^2 \le \|u\|^2 .$$

Moreover,

$$\|u - P_K(u)\|^2 = \sum_{\substack{k\in\mathbb{Z}^d \\ |k|\ge K}} |\langle \varphi_k, u\rangle|^2 ,$$

which converges as tail of converging sequence.

3. No proof.

4. For $d = 1$. If $u \in H^s(\mathbb{T})$. Then:

$$\begin{aligned}
\langle \varphi_k, u \rangle_{H^s} &= \sum_{p \leq s} \int_0^1 D^p u D^p {\varphi_k}^* dx \\
&= \sum_{p \leq s} (-1)^d \int_0^1 u D^{2p} {\varphi_k}^* dx \\
&= \sum_{p \leq s} (-1)^d (2\pi i k)^{2p} \int_0^1 u {\varphi_k}^* dx \\
&= \sum_{p \leq s} (-1)^d (2\pi i k)^{2p} \langle \varphi_k, u \rangle_{L^2},
\end{aligned}$$

which implies the coefficients are the same. Hence,

$$\begin{aligned}
\|u - P_K(u)\|_{H^s}^2 &= \sum_{\substack{k \in \mathbb{Z}^d \\ |k| \geq K}} |\hat{u}_k|^2 \|\varphi_k\|_{H^s}^2 \\
&= \sum_{\substack{k \in \mathbb{Z}^d \\ |k| \geq K}} |\hat{u}_k|^2 (2\pi)^2 \sum_{p \leq s} k^{2p} \\
&\geq (2\pi)^2 \sum_{p \leq s} K^{2p} \sum_{\substack{k \in \mathbb{Z}^d \\ |k| \geq K}} |\hat{u}_k|^2 \\
&= (2\pi)^2 \sum_{p \leq s} K^{2p} \|u - P_K(u)\|_{H^0}^2 \\
&\geq C K^{2s} \|u - P_K(u)\|_{H^0}^2 .
\end{aligned}$$

This implies the claim.

5. Using Sobolev inequality for $r \gneqq \frac{d}{2}$:

$$\|u - P_K(u)\|_{L^\infty} \leq C \|u - P_K(u)\|_{H^r} \leq C K^{r-s} \|u\|_{H^s} . \qquad \square$$

Point 3) implies the so called spectral convergence: as long as the function is smooth enough (in Sobolev norms), the convergence is faster than any polynomial.

4.1.2 Discrete Fourier transform

Remember, we would like compute the Fourier coefficients of a function $u(t) \in L^2(\mathbb{T}^d)^d$:

$$\hat{u}_k(t) := \langle \varphi_k, u(\cdot, t) \rangle = \int_0^1 \varphi_k(x) dx \in \mathbb{C}^d, \quad \forall k \in \mathbb{Z}^d, t \in [0, T]. \tag{4.7}$$

On a computer, it is common to introduce the discretization $\mathbb{T}_N^d$ of the torus $\mathbb{T}^d$ as $\mathbb{T}_N^d = \frac{1}{N}(\mathbb{Z}/N\mathbb{Z})^d$, for $N = 2K+1$ and evaluate the integral at this points:

$$\tilde{u}_k := \sum_{x\in\mathbb{T}_N^d} u(x)\varphi_k(x) \in \mathbb{C}^d, \quad \forall k \in \mathbb{Z}^d, t \in [0,T]. \tag{4.8}$$

We will call $\tilde{u}_k$ the *discrete Fourier coefficients.* Clearly, in general, $\tilde{u}_k \neq \hat{u}_k$. However, one can quantify the error made by this discretization.

Lemma 4.7 *We have:*

$$\tilde{u}_k - \hat{u}_k = \sum_{l\in\mathbb{Z}^d\setminus\{0\}} \hat{u}_{k+lN}, \quad \forall\,|k| \leq K \tag{4.9}$$

This error is called aliasing error*.*

Definition 4.8 *We define the* pseudo-spectral approximation*:*

$$\Psi_K(u) := \sum_{\substack{k\in\mathbb{Z}^d\\|k|\leq K}} \tilde{u}_k\varphi_k.$$

Then, we have the following:

$$\|u - \Psi_K(u)\|_{H^s} := \underbrace{\|u - P_K(u)\|_{H^s}}_{=:T_K^s} + \underbrace{\|P_K(u) - \Psi_K(u)\|_{H^s}}_{=:A_K^s}, \quad \forall s > 0$$

Since 4.9 involves only high modes of u, one can prove that the aliasing error is similar to the truncation error.

We have

$$A_K^s \leq CT_K^s. \tag{4.10}$$

Namely, the spectral convergence is retained when using discrete Fourier transform.

4.1.3 Spectral approximation on NSE

Our goal is to approximate the NSE. We will consider the formulation (we set zero forcing):

$$u_t + \operatorname{div}(u\otimes u) + \nabla p = \nu\Delta u, \tag{4.11a}$$

$$\operatorname{div} u = 0, \tag{4.11b}$$

or, applying the Leray projection (notice, in periodic settings $[\Delta, \mathbb{P}] \equiv 0$):

$$u_t + \mathbb{P}[\mathrm{div}(u \otimes u)] + \nu \Delta u.$$

Write a solution $u(t) = \sum_{k \in \mathbb{Z}^d} \hat{u}_k(t) \varphi_k$. Notice:

- $u_t = \sum_{k \in \mathbb{Z}^d} (\hat{u}_k)_t \varphi_k$;
- $\Delta u = \sum_{k \in \mathbb{Z}^d} \hat{u}_k \Delta \varphi_k = \sum_{k \in \mathbb{Z}^d} - (2\pi |k|)^2 \hat{u}_k \varphi_k$;
- $\mathbb{P} u = \sum_{k \in \mathbb{Z}^d} \hat{u}_k \mathbb{P} \varphi_k = \sum_{k \in \mathbb{Z}^d} \left(\hat{u}_k - \frac{\hat{u}_k \cdot k}{|k|^2} k \right) \varphi_k$;
- $\nabla f = \sum_{k \in \mathbb{Z}^d} \hat{f}_k \nabla \varphi_k = \sum_{k \in \mathbb{Z}^d} 2\pi i k \hat{f}_k \varphi_k$;
- $\mathrm{div}\, u = \sum_{k \in \mathbb{Z}^d} \hat{u}_k \,\mathrm{div}\, \varphi_k = \sum_{k \in \mathbb{Z}^d} 2\pi i k \cdot \hat{u}_k \varphi_k$.

Inserting the expansion into 4.11 and comparing component-wise:

$$(\hat{u}_k)_t(t) + \hat{B}_k(t) + ik\hat{p}_k(t) = -\nu k^2 \hat{u}_k(t) \tag{4.12}$$

$$ik \cdot \hat{u}_k = 0 \tag{4.13}$$

or, equivalently, using 4.1.3:

$$(\hat{u}_k)_t(t) + \left(\hat{B}_k(t) - \frac{\hat{B}_k(t) \cdot k}{|k|^2} k \right) = -\nu k^2 \hat{u}_k(t) \tag{4.14}$$

where $\{\hat{B}_k\}_k$ is the Fourier spectrum of $u \otimes u$, i.e. $\hat{B}_k = (\widehat{u \otimes u})_k$. Notice that the second equation is easier to treat, since the pressure and divergence-free constraint are completely removed from the equations. As long as we are able to compute the spectrum of $u \otimes u$, equation amounts to a infinitely dimensional ODE, which can be easily approximated using Fourier truncation and a standard ODE integrator.

4.1.4 Nonlinear term

We start by a simple computation:

$$\begin{aligned} u \otimes u = \left(\sum_{k \in \mathbb{Z}^d} \hat{u}_k \varphi_k \right) \otimes \left(\sum_{k \in \mathbb{Z}^d} \hat{u}_k \varphi_k \right) &= \sum_{k \in \mathbb{Z}^d} (\hat{u}_k \otimes \hat{u}_{k'}) \varphi_{k+k'} \\ &= \sum_{\substack{k \in \mathbb{Z}^d \\ j+l=k}} (\hat{u}_j \otimes \hat{u}_l) \varphi_k \end{aligned}$$

i.e.

$$\widehat{(u \otimes u)}_k = \sum_{j+l=k} \hat{u}_j \otimes \hat{u}_l, \tag{4.15}$$

and the same holds for a discrete approximation $P_K(u)$:

$$(P_K(\widehat{u) \otimes P}_K(u))_k = \sum_{j+l=k} \hat{u}_j \otimes \hat{u}_l, \quad \forall k \in \mathbb{Z}^d. \tag{4.16}$$

Notice that, in general, new non-zero modes are introduced. In practice, this introduces a global coupling of the modes in the ODE (4.12) and (4.14). While it is still possible to to solve those ODEs, it is unpractical to do so. Often a *pseudo spectral approximation* is used to compute $(P_K(\widehat{u) \otimes P}_K(u))_k$.

4.1.5 Pseudo spectral technique

The idea of pseudo spectral approximations is the following:

- differentiation/Leray projection is easy to do in Fourier space, and amounts to multiplication by some factor depending on the mode k;
- products are easy in time space but amount to convolution in Fourier space, therefore products are performed in Fourier space;
- sums are always easy to compute and are performed in either space.

Algorithm 1 Pseudo-spectral method for nonlinear term.

Input: Coefficients $\hat{u}_k$ for $|k| < K$
Output: Coefficients $\widetilde{w}_k \approx \widehat{(u \otimes u)}_k$ for $|k| < K$
  Compute the value at grid points $u(x)$ using DFT^{-1}
  Compute the product $w(x) = u(x) \otimes u(x)$
  Compute the Fourier coefficients $\tilde{w}_k$ of w using DFT

The complexity of this algorithm is the same as the one of a discrete Fourier transform. Generally, a discrete Fourier transform can be performed in $O(K \log(K))$ operations, therefore this algorithm is relatively cheap.

Notice, however, that, in general, $\widetilde{w}_k \neq \widehat{w}_k$.

Theorem 4.9 *The pseudo-spectral technique introduces the error:*

$$\widetilde{w}_k - \widehat{w}_k = \sum_{\substack{k'+k''=k+lN \\ |k'|,|k''| \leq K}} \widehat{u}_{k'} \otimes \widehat{u}_{k''},$$

for $|l| \leq 1, l \in \mathbb{Z}^d \setminus \{0\}$.

Proof

$$\begin{aligned}
\widetilde{w}_k &= \sum_{x\in\mathbb{T}_N^d} w(x)\varphi_{-k} \\
&= \sum_{x\in\mathbb{T}_N^d} u(x)\otimes u(x)\varphi_{-k} \\
&= \sum_{x\in\mathbb{T}_N^d} \left(\sum_{k'\in\mathbb{Z}^d} \hat{u}_{k'}\varphi_{k'}\right)\otimes\left(\sum_{k''\in\mathbb{Z}^d} \hat{u}_{k''}\varphi_{k''}\right)\varphi_{-k} \\
&= \sum_{\substack{x\in\mathbb{T}_N^d \\ k',k''\in\mathbb{Z}^d}} (\hat{u}_{k'}\otimes\hat{u}_{k''})\varphi_{k'}\varphi_{k''}\varphi_{-k} \\
&= \sum_{k',k''\in\mathbb{Z}^d} (\hat{u}_{k'}\otimes\hat{u}_{k''}) \sum_{x\in\mathbb{T}_N^d} \varphi_{k'+k''-k} \\
&= \sum_{k',k''\in\mathbb{Z}^d} (\hat{u}_{k'}\otimes\hat{u}_{k''}) \sum_{x\in\mathbb{T}_N^d} \varphi_{k'+k''-k} \\
&= \sum_{\substack{l,k',k''\in\mathbb{Z}^d, l\neq 0 \\ k'+k''=k+lN}} (\hat{u}_{k'}\otimes\hat{u}_{k''}).
\end{aligned}$$

Here, we used:

$$\sum_{x\in\mathbb{T}_N^d} \varphi_k(x) = \begin{cases} 1 & k = lN, l\in\mathbb{Z}^d \\ 0 & \text{otherwise} \end{cases}.$$

Notice that, if $|k|, |k'|, |k''| \leq K$, we have $|k' + k'' - k| < 3K = \frac{3}{2}(N-1)$. Hence $|l| < 1$. □

This error is also named *aliasing* error. Notice the similarity with the aliasing error above. This error may become quite visible when approximating the NSE, leading to temporal instabilities and spurious oscillations. Even if in practice such errors decays fast as the number of modes is increased, one generally seeks to remove this error to improve the stability of the scheme.

4.1.6 De-aliasing technique

The aliasing effect of the non-linear term can be removed with a simple modification of the pseudo-spectral methods, with cost-effective techniques.

Zero-padding

The simplest method to *completely* remove aliasing errors is to compute products using the pseudo-spectral technique after extending the number of modes by setting them to zero. This method is called 3/2-rule (or 2/3-rule). The name will become clearer later.

The idea is based on: let $K' > K$ then

$$u = \sum_{k \in \mathbb{Z}^d} \hat{u}_k \varphi_k = \sum_{k \in \mathbb{Z}^d} \hat{u}'_k \varphi_k \tag{4.17}$$

where

$$\hat{u}'_k = \begin{cases} \hat{u}_k & |k| < K \\ 0 & \text{otherwise} \end{cases}, \forall\, |k| < K'. \tag{4.18}$$

Algorithm 2 3/2-rule for de-aliasing nonlinear term.

Input: Coefficients $\hat{u}_k$ for $|k| < K$
Output: Coefficients $\widetilde{w}_k \approx (\widehat{u \otimes u})_k$ for $|k| < K$
Extend the spectrum of $\hat{u}'_k$ to size K' with zeros
Compute the value at grid points $u(x)$ using DFT^{-1}
Compute the product $w(x) = u(x) \otimes u(x)$
Compute the Fourier coefficients $\tilde{w}'_k$ of w using DFT
Compute $\tilde{w}$ by reducing the spectrum removing all modes $> K$.

Proposition 4.10 *For $K' > \frac{3}{2}(K - \frac{1}{2})$ the aliasing error is completely removed, i.e. $\tilde{w}'_k - \hat{w}'_k, \forall\, |k| < K$.*

Proof We have

$$\widetilde{w}'_k - \widehat{w}'_k = \sum_{\substack{k'+k''=k+lN' \\ |k'|,|k''| \le K'}} \widehat{u}_{k'} \otimes \widehat{u}_{k''},$$

but since $\frac{2}{3}(1 + K') > K$, $|k' + k'' - k| < 3K < 2(1 + K') = N'$. □

It is also possible to consider an improved de-aliasing technique by considering a smooth cutoff of high frequencies, i.e. $\hat{u}'_k = \sigma_k \hat{u}_k$ for

$$\sigma_k = \begin{cases} 1 & |k| < \frac{2}{3}K \\ 0 & |k| > K \end{cases}, \forall\, |k| < K' \tag{4.19}$$

and smooth between $\frac{2}{3}K$ and K.

Frequency shift

The general idea of frequency shift is to compute the values of the function at $u(x+\Delta)$, where Δ is a small perturbation. This has the benefit of not introducing extra memory requirement, but is generally more expensive and less used than the zero-padding technique.

4.1.7 Euler equations and spectral viscosity

If $\nu = 0$ and the solution u of the Euler equations lacks smoothness, the appearance of spurious oscillations (also known as Gibbs phenomenon) in solutions approximated with spectral methods may pollute the result. In order to mitigate such effects, it is possible to introduce a small numerical viscosity that becomes smaller as the number of modes is increased. This type of numerical viscosity is termed spectral viscosity, since it is generally chosen to be smaller for higher spectral modes. The Euler equations with spectral viscosity becomes:

$$(\hat{u}_k)_t + \hat{B}_k = SV[\hat{u}_k]. \tag{4.20}$$

An example of spectral viscosity [BT15] of order r, for $r \geq 1$ is:

$$SV[\hat{u}_k] = -K\sigma\left(\frac{|k|}{K}\right)\hat{u}_k, \quad \sigma(x) = \max\left\{0, |x|^{2r} - \frac{1}{N}\right\}. \tag{4.21}$$

Notice:

$$\left\|\sum_{k\leq K} SV[\hat{u}_k]\varphi_k\right\|_{H^s} \leq CK^{1-(s-r)(1-\frac{1}{2r})}\left\|\sum_{k\leq K}\hat{u}_k\varphi_k\right\|_{H^r}. \tag{4.22}$$

4.1.8 Convergence of spectral methods for NSE

One can prove that, in the semi-discrete case, the spectral methods converge to the solutions of NSE, provided sufficient smoothness of the solutions is available. The convergence is spectral. For instance, we have the following theorem (pf. in [BT15]).

Theorem 4.11 (Convergence of spectral methods) *Let $\nu = 0$, and consider (4.20) with $\frac{2}{3}$-de-aliasing rule. Let $u \in L^\infty([0,T], C^{1+\alpha}(\mathbb{T}))$ be a solution of the Euler equations with initial data u_0. Then. the approximated solution u_K, with $K > 0$ converges to u in $L_t^\infty L_x^2$. Moreover,*

$$\|u_K(t) - u\|_{L^2} < Ce^{2\|u\|_{L^1(W^{1,\infty})}}\left(\frac{1}{K^{2s}}\|u_0\|_{H^s}^2 + \frac{1}{K^{s-1-\frac{d}{2}}}\|u\|_{L^\infty H^s}\right). \tag{4.23}$$

4.1.9 Notes on implementation and complexity

In this section, we give a few remarks on spectral methods for NSE and EE.

In general a DFT can be done in $O(N \log(N))$, N being the number of modes. A fast algorithm is the Cooley-Tukey algorithm, which is available for 2^N modes. In general, it is possible to have efficient FFTs for number of modes that are factorizable with small primes (e.g. $N = 2^a 3^b 5^c 7^d$). The prime factorization of the modes has a large impact on the efficiency of the DFT. When choosing a number of modes for the de-aliasing rule, it is convenient to pad the spectrum to a power of 2 or to a number with small prime factors.

In this context, the resulting Fourier transform is actually a real Fourier transform (i.e. the time space is real). This implies that the Fourier coefficients $\hat{u}_k \in \mathbb{C}^d$ satisfy the relation $\hat{u}_k = \hat{u}_{-k}{}^*$ and therefore is only necessary to store and compute half of the spectrum.

How the DFT is implemented and how the data is stored in the memory has a large impact on the performance of the numerical method.

The total complexity of the scheme, assuming we discretize (4.20) in time using a standard explicit Runge-Kutta scheme (which implies that each time-step consists of a small, constant number of DFT), is $O(MN \log(N))$, where M is the number of steps chosen (cf. next section to more in-depth discussion on the choice of M).

A multi-dimensional DFT can be efficiently performed in parallel, considering the tensor-structure of the formula for the coefficients:

$$\hat{u}_k = \sum_{x_1 \in h\mathbb{Z}/K\mathbb{Z}} e^{-ik_1 x_1} \cdots \sum_{x_d \in h\mathbb{Z}/K\mathbb{Z}} e^{-ik_d x_d} u(x). \tag{4.24}$$

4.2 Time-stepping

So far, we completely ignored the temporal discretization of the NSE equations. If we consider the periodic setting, the absence of boundaries makes it easy to discretize the equations in time.

4.2.1 Explicit time-stepping

For a fully explicit discretization, one could use any flavour of Runge-Kutta method. Rewriting equation (4.20), one obtains:

$$u_t = L(u), \tag{4.25}$$

and, after spatial discretization, one obtains a system of ODEs, which can be integrated in time using any form of Runge-Kutta method. Of particular interest are the so called Strong Stability Preserving Runge-Kutta (SSP RK) methods [GST01], which are often employed when treating systems of conservation laws. This class methods have strong stability properties for linear ODEs and may effectively be used fo approximate NSE, especially when $\nu << 1$. In this case, a time stepping restriction has to be imposed in order to ensure stability of the scheme. As an example, a second order SSP method is:

$$\begin{aligned} u^1 &= u^n + \Delta_t L(u^n), \\ u^{n+1} &= \frac{1}{2}(u^n + u^1 + \Delta t L(u^1)). \end{aligned}$$

A third order method is:

$$\begin{aligned} u^1 &= u^n + \Delta_t L(u^n), \\ u^2 &= \frac{1}{4}(3u^n + u^1 + \Delta t L(u^1)), \\ u^{n+1} &= \frac{1}{3}(u^n + 2u^2 + 2\Delta t L(u^2)). \end{aligned}$$

4.2.2 Implicit time-stepping

It is well-known that explicit time-integrations suffer of conditional stability and necessitate of time-stepping restrictions in order to ensure stability.

Sometimes, one wishes to approximate long time-behaviour of NSE, $T >> 1$. Additionally, one may want to compute steady-state solutions to NSE, by means of pseudo-time stepping (i.e. solving (4.11) for $T = \infty$). In this cases, it is often advisable to employ implicit or semi-implicit time-stepping procedures.

It is generally common to discretize the time considering each term individually: the viscosity term is generally treated implicitly (avoiding restrictive time-step sizes), whilst the nonlinear term is treated explicitly (avoiding the necessity of solving non-linear systems).

Viscous term

Consider, for the moment, the momentum equation 4.11a without nonlinear term. Doing so, one obtains a simple system of decoupled heat equations:

$$u_t = \nu \Delta u \tag{4.26}$$

For this equation, it is well known that implicit methods are stable and very effective in approximating the equation. The solution of the system arising from the implicit discretization is generally easy to perform and many efficient methods exist. Moreover, the spectral methods induce a completely decoupled system of equations, which is trivial to solve.

In general a good (low-order) way to approximate the heat equation (4.26) is to employ implicit mid-point (which, in this setting, is equal to Crank-Nicolson discretization) or the implicit Euler method:

$$\begin{aligned} \frac{u^{n+1}-u^n}{\Delta t} &= \frac{\nu}{2}\Delta(u^{n+1}+u^n) \\ \frac{u^{n+1}-u^n}{\Delta t} &= \nu\Delta u^{n+1}. \end{aligned}$$

Notice how both schemes are unconditionally L^2-stable (exercise: multiply by $u^{n+1}+u^n$ resp. u^{n+1}).

The nonlinear term

We now consider the nonlinear term alone, and remove the pressure from the equation, we obtain the equation:

$$u_t = (u \cdot \nabla)u) \tag{4.27}$$

It is possible to discretize the system implicitly, e.g. with a second order discretization:

$$\begin{aligned} \frac{u^{n+1}-u^n}{\Delta t} &= \frac{1}{2}((u^{n+1}\cdot\nabla)^{n+1}) + (u^n\cdot\nabla)^n)), \\ \frac{u^{n+1}-u^n}{\Delta t} &= \frac{1}{4}((u^{n+1}+u^n)\cdot\nabla)(u^{n+1}+u^n)). \end{aligned}$$

The resulting system (after application of some spatial discretization) becomes a nonlinear system of equations, which has to be solved with expensive iterative methods (e.g. a fixed point iteration). On the other hand, one could explicitly discretize the equation. The advantage of discretizing the term explicitly is that no system has to be solved (we will see the benefits of explicit evaluation of this term in the finite difference section).

4.3 Finite difference projection methods

Whilst spectral methods are an efficient, accurate, and easy to implement approximations for PDEs, it is obvious from the discussion above that the treatment of more complex flows, such as a flow with boundaries, or a irregular flows, is very difficult, if not impossible, using solely spectral methods. An efficient and simple alternative to spectral methods are finite difference and finite volume methods.

4.3.1 Overview of finite difference methods

Again, let us rewrite the Navier-Stokes equations in velocity-pressure form augmented with boundary conditions. For $\nu \geq 0$ and $\Omega \subseteq \mathbb{R}^d$:

$$u_t + \operatorname{div}(u \otimes u) + \nabla p = \nu \Delta u, \quad \text{on } \Omega \times [0, T] \tag{4.28}$$

$$\operatorname{div} u = 0, \tag{4.29}$$

$$u|_{\partial\Omega} = 0, \quad \text{on } \partial\Omega \times [0, T] \tag{4.30}$$

$$u(\cdot, 0) = u_0. \tag{4.31}$$

In the case of Euler equations, we will impose the boundary condition

$$u|_{\partial\Omega} \cdot \eta = 0, \quad \text{on } \partial\Omega \times [0, T] \tag{4.32}$$

where η is the outward unit normal vector.

In this section, we will consider, for simplicity, the 2d Navier-Stokes on the domain $[0, 1]^2 =: \Omega$. We fix a number $\mathbb{N} \ni N > 0$ and let $h := 1/N$. With this domain, we discretize the solution using piecewise constant functions, and write, by abuse of notation, $u^n_{i,j} := u(x_{i,j}, t^n)$, where $x_{i,j} = (ih + h/2, jh+h/2) \in \Omega$ and $t^n = n\Delta t$ for a fixed $\Delta t > 0$. On dimensions $d \neq 2$ all definitions remain analogous. The idea of finite difference approximations is to replace all spatial and temporal derivatives with difference quotients.

In the previous sections, we have seen that the NSE equation can be rewritten in many different equivalent formulations (notably the velocity-pressure form and the vorticity form). Moreover, we have seen how the nonlinear term, due to divergence free constraint, can be written in many different, equivalent forms. However, applying a finite difference scheme to different equations may yeld a completely different discretization with different properties.

Example 4.12 *In 1d, let us consider the term $(u^2)_x = 2uu_x$. If we approximate the first term and the second term directly using central differences:*

$$(u^2)_x \approx \frac{u^2_{j+1} - u^2_{j-1}}{2h} \neq 2u_j \frac{u_{j+1} - u_{j-1}}{2h} \approx 2uu_k.$$

A notable issue with finite differences is that there is no chain rule.

As justification for the next section, we will consider some simplification to NSE.

Every form of NSE we have seen contains a nonlinear term, similar to e.g. $\operatorname{div}(u \otimes u), (u \cdot \nabla)u, (u \cdot \nabla)\omega + (\omega \cdot \nabla)u$. A prototypical equation for this term is the linear advection equation in 1D:

$$u_t = a u_x.$$

A simple way to discretize this equation with finite differences is using forward in time/central in space schemes:

$$\frac{u^{n+1} - u^n}{\Delta t} = a \frac{u_{j+1} - u_{j-1}}{2h}.$$

However, applying stability analysis to this equation (e.g. Von Neumann stability analysis), clearly shows that this discretization is unconditionally unstable. A common solution to this problem is to *upwind* the discretization, based on the sign of the advection velocity a. The upwind scheme is:

$$\frac{u^{n+1} - u^n}{\Delta t} = \begin{cases} a\frac{u_j - u_{j-1}}{h} & a > 0 \\ a\frac{u_{j+1} - u_j}{h} & a \leq 0. \end{cases}$$

If we apply Von Neumann stability analysis to this equation, we obtain the stability condition:

$$\Delta t \leq \frac{\Delta x}{a}. \tag{4.33}$$

This condition is generally termed CFL condition (after Courant-Friedrichs-Lewy). In the context of Navier stokes equations, one could say that $a = \|u\|_{L^\infty}$. Assuming the velocity remains bounded, one has the condition $\Delta t \simeq \Delta x$.

On the other hand, if one considers the NSE without nonlinearity, and considers the viscous term alone, one has the heat equation:

$$u_t = \nu \Delta u$$

A common approximation for this equation is:

$$\frac{u^{n+1} - u^n}{\Delta t} = \nu \frac{u_{j+1} + 2u_j + u_{j-1}}{h^2}.$$

Applying stability analysis to this equation, one obtains the much more restrictive:

$$\Delta t \leq \frac{1}{2} \frac{\Delta x^2}{\nu}, \tag{4.34}$$

which, in turn, implies, $\Delta t \simeq \Delta x^2$. It is therefore clear that, at low Reynolds number ($\nu >> 1$), this discretization quickly becomes unaffordable.

For this reason, the viscosity is generally treated implicitly.

4.3.2 Naive implementation

Let we write the NSE in the projected form:

$$u_t + \mathbb{P} \operatorname{div}(u \otimes u) = \nu \mathbb{P} \Delta u, \quad \text{on } \Omega \times [0, T] \tag{4.35}$$

$$u|_{\partial\Omega} = 0, \quad \text{on } \partial\Omega \times [0, T] \tag{4.36}$$

$$u(\cdot, 0) = u_0. \tag{4.37}$$

A naive scheme approximating the NSE would approximate each operator using finite differences and using, e.g. Crank-Nicolson time integration:

$$\frac{u^{n+1} - u^2}{\Delta t} + \mathbb{P}^h \operatorname{div}^h(u^{n+1/2} \otimes u^{n+1/2}) = \nu \mathbb{P}^h \Delta^h u^{n+1/2}, \tag{4.38}$$

where

- $u^{n+1/2} := \frac{u^{n+1} - u^n}{2}$, is the time average;
- $(D_x^h u)_{i,j} := \frac{u_{i+1,j} - u_{i-1,j}}{2h}$, is the central difference;
- $(D_x^{h,-} u)_{i,j} := \frac{u_{i,j} - u_{i-1,j}}{h}$, is the backward difference;
- $(D_x^{h,+} u)_{i,j} := \frac{u_{i+1,j} - u_{i,j}}{h}$, is the forward difference;
- $\nabla^h u := \left(D_x^h u, D_y^h u\right)^\perp$, is the central gradient;
- $\operatorname{div}^h u := D_x^h u_1 + D_y^h u_2$, is the central divergence;
- $\Delta^h u := D_x^{h,-} D_x^{h,+} u + D_y^{h,-} D_y^{h,+} u$, is the standard 5-points Laplace operator;
- $\mathbb{P}^h u := u - \nabla^h (\Delta^h)^{-1} \operatorname{div}^h u$, is the standard 5-points Laplace operator;

In principle, it is possible to solve the big non-linear system arising from this discretization. Note, however, that there are many problems with this approach. Notice

- the system (4.38) is unstable, since we used a centered approximation of $\operatorname{div}(u \otimes u)$;
- since computing $\mathbb{P}^h$ involves solving a linear system, the solution of (4.38) is very expensive, since one has to solve a system inside a non-linear solver.
- since $\operatorname{div}^h \nabla^h \neq \Delta^h$, in general, we have that $Div^h \mathbb{P}^h u \neq 0$, unless a different type of finite difference approximations is used;
- the treatment of boundaries for this system (in particular for $\mathbb{P}^h$ is difficult.

We seek methods to improve the stability and efficiency of the finite difference approximation. This will be done by rethinking the way we treat the spatial discretization, as well as the way we treat the temporal discretization.

4.3.3 Projection methods

Projection methods where introduced in [Cho68, **?**] if the '60. Improved versions can be found e.g. in [Van86, ABS96, BCG89]. The basic idea is still widely used today in many common CFD solvers. The projection methods are also called *fractional step methods* or *pressure correction methods*. The basis of projection methods is operator splitting in time, to decouple the evolution phase (where viscosity and convection are considered), from the incompressibility phase (equivalently the pressure phase). This greatly reduces the cost of the solution to a solution of an advection-diffusion equation and a Poisson equation.

The basic idea is the following:

Step 1, prediction : compute a prediction of the velocity u^{n+1}, called $u^{*,n+1}$, using the advection diffusion equation (4.28), and without enforcing the incompressibility condition (4.29). The pressure is treated explicitly, using an old (best-possible) value for the pressure or completely removing the pressure term. Without spatial discretization, this results in:

$$\frac{u^{*,n+1} - u^n}{\Delta t} = \nu\Delta\left(u^n\right) + [(u^n \cdot \nabla)u^n]. \tag{4.39}$$

Step 2, correction (or projection) : using the Hodge-Helmholtz decomposition, the velocity (a priori non divergence-free) is projected onto the space of divergence-free velocities, whilst the gradient part is used to correct the values for the pressure:

$$u^{n+1} = \mathbb{P}u^{*,n+1}. \tag{4.40}$$

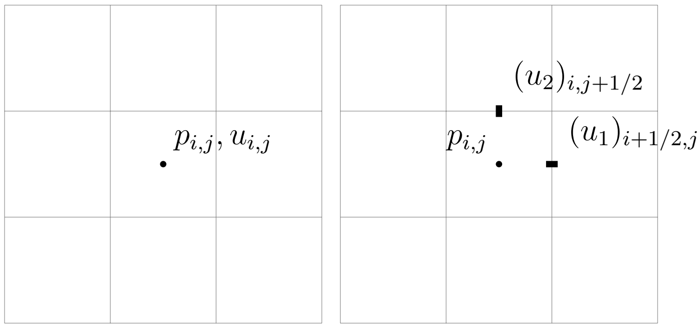
Remark 4.13 *As we have seen, it is common to treat the advection part explicitly and the viscosity part implicitly. This results in the prediction system* (4.39) *being a linear system of equations, which can be seen as a heat equation with forcing. Therefore, it is possible to efficiently solve this system using common linear algebra techniques. The correction phase involves the approximation of the Leray projection, which may be efficiently done with the solution of a Poisson equation.*

Remark 4.14 *The treatment of the boundaries for the projection method is delicate. In the prediction phase* (4.39) *we enforce Dirichlet boundaries for the velocity* ($u^{*,n+1}|_{\partial\Omega} = 0$)*. However, in the projection phase, we use the Hodge-Helmholtz decomposition, which imposes the no-slip boundary condition* $u^{n+1}|_{\partial\Omega} \cdot \eta = 0$*. This results in projection methods having larger errors on the boundary. This has the additional consequence that the pressure computed by means of the projection method has no real physical meaning and should only be considered as a Lagrange multiplier.*

4.3.4 MAC Scheme

The MAC scheme (Marker-and-cell), introduced in [HW65] for moving surfaces, is based on the observation that the components of the velocity need not be placed at the cell centers, i.e. at the same coordinates as the pressure. If the coordinates for the points where the components of the velocity are chosen appropriately, then many finite difference methods become more stable and with additional properties.

The idea is that the velocity is stored only in its normal component, and it is placed at the edge of the cells. For instance, on Cartesian grids, the first component u_1 of the velocity is located at the point $x_{i+1/2,j} := (hi, hj + 1/2h)$ and is denoted $(u_1)_{i+1/2,j}$. Similarly, the second component of the velocity is denoted $(u_2)_{i,j+1/2}$ and is placed at the center of horizontal edges.



(a) Collocated grid (b) MAC grid

Figure 4.1: On the left, a collocated grid, where all quantities are positioned at cell centers, on the right, the MAC positioning, where the velocity components are placed at the edges.

With a velocity defined with the MAC scheme, it is easy to compute the divergence as a central difference located at the cell centers:

$$\operatorname{div} u \approx \frac{(u_1)_{i+1/2,j} - (u_1)_{i-1/2,j}}{h} + \frac{(u_2)_{i,j+1/2} - (u_2)_{i,j-1/2}}{h} =: (\operatorname{div}^h u)_{i,j}$$

and, similarly, the gradient of cell centered quantities, can be defined as an adjoint operator on the velocity coordinates:

$$\nabla u \approx \begin{pmatrix} \frac{u_{i+1,j}-u_{i,j}}{h} \\ \frac{u_{i,j+1}-u_{i,j}}{h} \end{pmatrix} =: (\nabla^h u)_{i,j}$$

such that the standard 5-points Laplace $\Delta^h = \operatorname{div}^h \nabla^h$. Is is then natural to define the Leray projection as: $\mathrm{Id} - \nabla^h (\Delta^h)^{-1} \operatorname{div}^h$ from cell centre to cell centers. The treatment of the nonlinearity is done in the following way:

$$\operatorname{div} u \otimes u \approx \begin{pmatrix} \frac{(\tilde{u}_1)^2_{i+1,j} - (\tilde{u}_1)^2_{i,j}}{h} + \frac{(\hat{u}_1)_{i,j+1/2}(\hat{u}_2)_{i,j+1/2} - (\hat{u}_1)_{i,j-1/2}(\hat{u}_2)_{i,j-1/2}}{h} \\ \frac{(\hat{u}_2)_{i+1/2,j}(\hat{u}_1)_{i+1/2,j} - (\hat{u}_2)_{i-1/2,j}(\hat{u}_1)_{i-1/2,j}}{h} + \frac{(\tilde{u}_2)^2_{i,j+1} - (\tilde{u}_2)^2_{i,j}}{h} \end{pmatrix},$$

where

$$(\tilde{u}_1)_{i,j} := \frac{(u_1)_{i+1/2,j} + (u_1)_{i-1/2,j}}{2}, (\tilde{u}_2)_{i,j} := \frac{(u_1)_{i,j+1/2} + (u_1)_{i,j-1/2}}{2},$$
$$(\hat{u}_1)_{i+1/2,j+1/2} := \frac{(u_1)_{i,j+1} + (u_1)_{i,j-1}}{2}, (\hat{u}_2)_{i+1/2,j+1/2} := \frac{(u_1)_{i,j+1} + (u_1)_{i,j-1}}{2}.$$

The viscosity is approximated using:

$$\Delta u \approx \frac{(u_1)_{i+3/2,j} + (u_1)_{i+1/2,j} + (u_1)_{i-1/2,j}}{2h} + \frac{(u_2)_{i,j+3/2} + (u_2)_{i,j+1/2} + (u_2)_{i,j-1/2}}{2h}.$$

4.3.5 Upwind BCG scheme

The following scheme is due to Bell, Colella and Glaz [BCG89], and is a second order finite difference method, which uses all the techniques seen so far. It exploits an implicit time-stepping for the treatment of the viscosity, a second order explicit reconstruction for the nonlinear term, and the pressure is decoupled from the momentum equation using a projection. The treatment of the nonlinearity borrows techniques from finite volume methods for conservation laws.

Time discretization and splitting

Using a projection method, the 2 steps of this scheme are:

- the prediction is done treating the viscosity implicitly, and using a second order in time reconstruction of the nonlinear term. An old value of the pressure is used, which is obtained in the correction step. The initial pressure at $n = 0$ may be set to zero. Given u^n, a prediction $u^{*,n+1}$ for u^{n+1} is given by:

$$\frac{u^{*,n+1} - u^n}{\Delta t} = \nabla p^{n-1/2} + \frac{\nu}{2}\Delta\left(u^{*,n+1} - u^n\right) + [(u \cdot \nabla)u]^{n+1/2}.$$

  The nonlinear term $[(u \cdot \nabla)u]^{n+1/2}$ is explicitly computed from u^n alone and will be specified later. Notice how this equation amounts to the solution of an heat equation, which is linear and can be done efficiently, provided the nonlinear term is available.

- in the correction step, the Hodge-Helmholtz decomposition is applied to $u^{*,n+1}$:

$$\begin{aligned} \frac{u^{*,n+1} - u^n}{\Delta t} = \frac{u^{n+1} - u^n}{\Delta t} + \nabla q, \quad \operatorname{div} u^{n+1} = 0 \\ \nabla p^{n+1/2} = \nabla q + \nabla p^{n-1/2}. \end{aligned}$$

Notice how many different flavours of projection can be used:

- velocity increment:

$$\begin{aligned} u^{*,n+1} = u^{n+1} + \Delta t \nabla q, \quad \operatorname{div} u^{n+1} = 0 \\ \nabla p^{n+1/2} = \nabla q + \nabla p^{n-1/2}. \end{aligned}$$

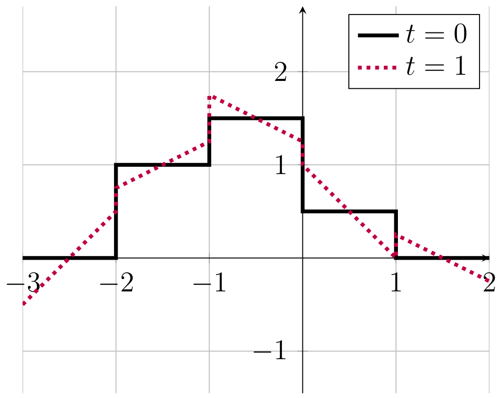


Figure 4.2: .

- pressure increment:

$$u^{*,n+1} = u^{n+1} + \Delta t \nabla q, \quad \operatorname{div} u^{n+1} = 0$$
$$p^{n+1/2} = q + p^{n-1/2}.$$

- pressure update:

$$u^{*,n+1} = u^{n+1} + \Delta t \nabla p^{n+1/2}, \quad \operatorname{div} u^{n+1} = 0$$

All those methods are equivalent without spatial discretization, but become different once all operators are discretized.

Space discretization

The viscous term can be approximated using a standard centered Laplace operator. The main difficulty arises from the nonlinear term. We want to approximate $(u \cdot \nabla)u$ at $t = t^{n+1/2}$ given the velocity $u^n_{i,j}$ at the cell centres and at time t^n. This is done in 6 steps:

1. *reconstruct* the derivatives u_x and u_y as piecewise constant functions inside each cell using centered differences, e.g. $(u_1)_x \approx \frac{u_{i+1,j} - u_{i-1,j}}{h}$. This, however, introduces new extreme points for u, which is not desirable, since this introduces unwanted oscillations in the solution. Therefore, we perform step 2:

2. *limiting*: the slopes reconstructed in this way are "limited", s.t. no new extrema is introduced:

$$(D_x u_1)_{i,j} := \operatorname{minmod}(D^h_x u_1, \operatorname{maxmod}(2D^{h,-}_x u_1, 2D^{h,+}_x u_1)),$$

where

$$\operatorname{minmod}(x,y) = \begin{cases} x & |x| < |y|\,, \\ y & |x| > |y|\,, \\ 0 & \operatorname{sign}(x) \neq \operatorname{sign}(y), \end{cases}$$

$$\operatorname{maxmod}(x,y) = \begin{cases} x & |x| > |y|\,, \\ y & |x| < |y|\,, \\ 0 & \operatorname{sign}(x) \neq \operatorname{sign}(y). \end{cases}$$

All other derivatives are obtained similarly.

3. *time derivatives*: the next step is to approximate the time derivative u_t. We approximate u_t at the cell centers using:

$$\begin{aligned} u_t &= \mathbb{P}(\nu\Delta u - (u\cdot\nabla)u) \approx \nu\Delta u - (u\cdot\nabla)u + \nabla p^{n-1/2}, \\ &= \nu\Delta u - \begin{pmatrix} (u_1)(u_1)_x + (u_2)(u_1)_y \\ (u_1)(u_2)_x + (u_2)(u_2)_y \end{pmatrix} + \nabla p^{n-1/2}, \end{aligned}$$

where we use the derivative $D_x u_1$ and $D_x u_2$ computed in the previous step (2) to approximate $(u_1)_x$ and $(u_2)_y$. For stability reasons, we use:

$$(\tilde{D}_y u_1)_{i,j} := \begin{cases} (D_x^- u_1)_{i,j} + \frac{1}{2}(1 - \frac{\Delta t}{h} v_{i,j})(D_y^h u_1)_{i,j}, & (u_2)_{i,j} \geq 0, \\ (D_x^- u_1)_{i,j} + \frac{1}{2}(1 - \frac{\Delta t}{h} v_{i,j})(D_y^h u_1)_{i,j}, & (u_2)_{i,j} < 0, \end{cases}$$

for the approximation of $(u_1)_y$ (and analogously for $(u_2)_x$). Let us call this approximation $D_t u$.

4. *extrapolation*: from the values of u and its derivatives at t^n and at the cell centres, we want to obtain the values of u at $t^{n+1/2}$ and located at the edges, this is done using Taylor approximation:

$$\begin{aligned} u^{L,n+1/2}_{i-1/2,j} &= u^n_{i,j} - \frac{h}{2}(D_x u)^n_{i,j} + \frac{\Delta t}{2}(D_t u)^n_{i,j} \\ u^{R,n+1/2}_{i+1/2,j} &= u^n_{i,j} + \frac{h}{2}(D_x u)^n_{i,j} + \frac{\Delta t}{2}(D_t u)^n_{i,j} \\ u^{B,n+1/2}_{i,j-1/2} &= u^n_{i,j} - \frac{h}{2}(D_y u)^n_{i,j} + \frac{\Delta t}{2}(D_t u)^n_{i,j} \\ u^{T,n+1/2}_{i,j+1/2} &= u^n_{i,j} + \frac{h}{2}(D_y u)^n_{i,j} + \frac{\Delta t}{2}(D_t u)^n_{i,j} \end{aligned}$$

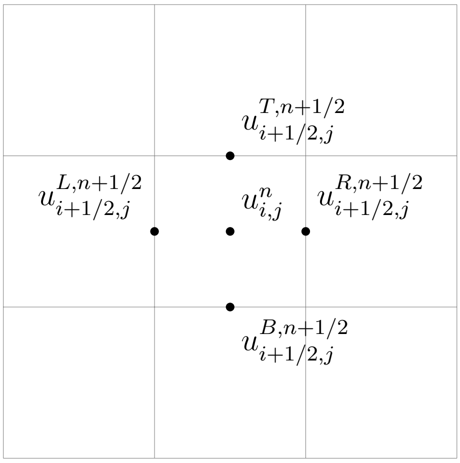


Figure 4.3: From the value of u at t^n at the cell centre, the value of u at $t^{n+1/2}$ at the edges is extrapolated using Taylor approximation. Four values (6 in 3D) are reconstructed and placed at the mid-point of the edge.

5. *solve Riemann problem*: at this stage, we have values of u at $t^{n+1/2}$ located at the edges (cf. Figure 4.3). For each edge there are two values, reconstructed from the two adjacent cells. Looking at the problem in a one-dimensional way (as example, for a vertical edge), one may view this as a Riemann problem, with the following data:

$$u_0(x) = H_{x_0}(x) := \begin{cases} u^L & x \leq x_0, \\ u^R & x > x_0, \end{cases} \tag{4.41}$$

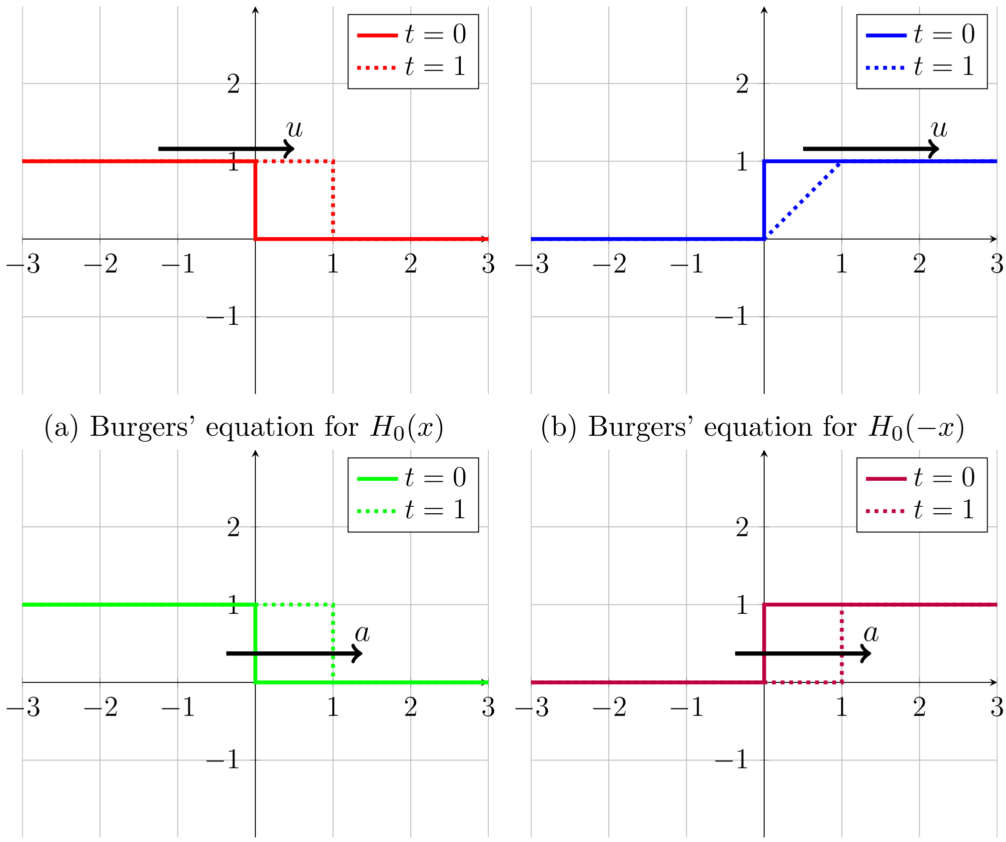
where x_0 is the x coordinate of the edge. If we assume that the velocity at this stage is already divergence-free and there is no viscosity in for this problem, we can view the governing equations as:

$$(u_1)_t = u_1(u_1)_x + u_2(u_1)_y, \tag{4.42}$$
$$(u_2)_t = u_1(u_2)_x + u_2(u_2)_y. \tag{4.43}$$

At the cells interface, we can view this problem as having the values of the velocity to be constant along the edge, e.g. $(u_2)_y = 0$ for a vertical edge. Therefore, we obtain the following equation:

$$(u_1)_t = u_1(u_1)_x \tag{4.44}$$
$$(u_2)_t = u_1(u_2)_x \tag{4.45}$$

Notice, how the first equation is the so called Burgers' equation, whilst the second equation is a linear advection equation with advection



(a) Burgers' equation for $H_0(x)$ (b) Burgers' equation for $H_0(-x)$

(c) Linear advection for $H_0(x)$ (d) Linear advection for $H_0(-x)$

Figure 4.4: Solutions of two types of Riemann problem encountered at the interface of the cell for two equations: Burgers' equation and linear advection equation. The choice of the velocity $u^{n+1/2}$ at the edges is dictated by the solution of the corresponding Riemann problem.

velocity u_1. The first equation is completely decoupled, whilst the second equation requires the solution of the first equation in order to retrieve the advection speed. Figure 4.4 shows all Riemann problems and solutions that occur when solving this decoupled system. This justifies the following reconstruction for the components (here only for a vertical edge, we drop temporal and spatial indices, since all quantities are evaluated at the same location):

$$u_1 := \begin{cases} u_1^L & u_1^L \geq 0, u_1^L + u_1^R \geq 0 \\ 0 & u_1^L u_1^R < 0 \\ u_1^R & \text{otherwise.} \end{cases} \qquad (4.46)$$

and

$$(u_2)_{i+1/2,j} := \begin{cases} u_2^L & (u_1)_{i+1/2,j} \geq 0 \\ \frac{u_2^L + u_2^R}{2} & (u_1)_{i+1/2,j} = 0 \\ u_2^R & (u_1)_{i+1/2,j} \leq 0. \end{cases} \quad . \tag{4.47}$$

The same reconstruction can be performed also for horizontal edges.

6. *compute the non-linear term*: at this stage, we have $u^{n+1/2}$ defined on edges of the cell. One can use finite differences to finally compute the non-linear term:

$$[(u \cdot \nabla)u]^{n+1/2} := \frac{(u_1)^{n+1/2}_{i+1/2,j} + (u_1)^{n+1/2}_{i+1/2,j}}{2} \frac{u^{n+1/2}_{i-1/2,j} + u^{n+1/2}_{i+1/2,j}}{h} + \frac{(u_2)^{n+1/2}_{i,j+1/2} + (u_2)^{n+1/2}_{i,j+1/2}}{2} \frac{u^{n+1/2}_{i,j+1/} - u^{n+1/2}_{i,j-1/2}}{h}$$

With this, the definition of the scheme is complete. The resulting scheme is numerically second order accurate in absence of boundaries.

A Results from Functional Analysis

In this appendix, we include some fundamental results from functional analysis that are used throughout these lecture notes. Most of these results can be found in any standard text on functional analysis and the theory of partial differential equations such as, for example, [Eva10, Chapter 5] and [Fol13, Chapter 4] but for the sake of completeness, we present them here.

A.1 Sobolev Inequalities

Theorem A.1 (Gagliardo-Nirenberg-Sobolev Inequality) *Let $d \in \mathbb{N}$, let $p \in [1, d)$, let $u \in \mathrm{C}_c^1(\mathbb{R}^d)$ and let $p^* = \frac{dp}{d-p}$. Then there exists some constant $C = C(p, d)$ such that*

$$\|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}.$$

Proof A detailed proof of this theorem can, e.g., be found in [Eva10, Section 5.6.1]. □

In the case of functions defined on bounded domains, we also have the following result:

Theorem A.2 (Poincaré Inequality) *Let $d \in \mathbb{N}$, let $p \in [1, d)$, let $\Omega \subset \mathbb{R}^d$ be a bounded, open set, let $u \in W_0^{1,p}(\Omega)$ and let $p^* = \frac{dp}{d-p}$. Then for each $q \in [1, p^*]$, there exists some constant $\bar{C} = \bar{C}(p, d, \Omega, q)$ such that*

$$\|u\|_{L^q(\Omega)} \leq \bar{C} \|\nabla u\|_{L^p(\Omega)}.$$

Proof A detailed proof of this theorem can, e.g., be found in [Eva10, Section 5.6.1]. □

Theorem A.3 (Gagliardo-Nirenberg Interpolation Inequality) *Let $d, m \in \mathbb{N}$, let $1 \leq q, r \leq \infty$, let $u\colon \mathbb{R}^d \to \mathbb{R}$ and let $\alpha \in \mathbb{R}$ and $j \in \mathbb{N}$ be such that*

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q},$$

and also

$$\frac{j}{m} \leq \alpha \leq 1.$$

Then there exists some constant $C = C(m, d, j, q, r, \alpha)$ such that

$$\|\nabla^j u\|_{L^p} \leq C\|\nabla^m u\|_{L^r}^{\alpha} \cdot \|u\|_{L^q}^{1-\alpha}.$$

We have the following two exceptional cases:

(i) If $1 < r < \infty$ and $m - j - \frac{d}{r} \in \mathbb{N}$, then the inequality only holds for $\frac{j}{m} \leq \alpha < 1$.

(ii) If $j = 0$, $rm < d$, $q = \infty$, one needs in addition that u tends to zero at infinity or $u \in L^{\widetilde{p}}(\mathbb{R}^d)$ for some finite $\widetilde{p} > 0$.

Proof This theorem and its proof can be found in [Nir59]. □

The following special case of Theorem A.3 is particularly useful and is used repeatedly in these lecture notes:

Theorem A.4 (Ladyzhenskaya's Inequality) *Let $d \in \{2, 3\}$, let $\Omega \subseteq \mathbb{R}^d$ be a Lipschitz domain and let $u\colon \Omega \to \mathbb{R}$ be a function that vanishes on the boundary $\partial\Omega$. Then there exists some constant $C = C(\Omega)$ such that*

$$d = 2 \implies \|u\|_{L^4} \leq C\|u\|_{L^2}^{\frac{1}{2}} \cdot \|\nabla u\|_{L^2}^{\frac{1}{2}},$$

$$d = 3 \implies \|u\|_{L^4} \leq C\|u\|_{L^2}^{\frac{1}{4}} \cdot \|\nabla u\|_{L^2}^{\frac{3}{4}}.$$

Proof This theorem and its proof can be found in [Lad58]. □

We conclude this section by stating a result on the Hölder continuity of certain weakly differentiable functions.

Theorem A.5 (Morrey's Inequality) *Let $d, p \in \mathbb{N}$ such that $n < p \leq \infty$, let $\gamma = 1 - \frac{d}{p}$ and let $u \in \mathrm{C}^1(\mathbb{R}^d)$. Then there exists some constant $C = C(d, p)$ such that*

$$\|u\|_{\mathrm{C}^{0,\gamma}} \leq C\|u\|_{W^{1,p}}.$$

Proof A detailed proof of this theorem can, e.g., be found in [Eva10, Section 5.6.2]. □

We remark that Theorem A.5 also holds for functions defined on a bounded domain.

A.2 Results on Compactness

We begin by defining the notion of *weak convergence* and *weak** convergence.

Definition A.6 (Weak Convergence) *Let V be a Banach space, let V^* denote its dual space, let $f \in V$ and let $\{f_m\}_{m\in\mathbb{N}} \subset V$ be a sequence with the property that for all linear functionals $\phi \in V^*$ it holds that*

$$\lim_{m\to\infty} \langle f_m, \phi\rangle = \langle f, \phi\rangle.$$

Then we say that the sequence $\{f_m\}_{m\in\mathbb{N}}$ converges in the weak sense *to f and we write $f_m \rightharpoonup f$.*

Remark A.7 *Consider the setting of the above definition, let $1 \leq p < \infty$ and let Ω be a set such that $V = L^p(\Omega)$. Then, $V^* = \big(L^p(\Omega)\big)^* = L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Thus, the sequence $\{f_m\}_{m\in\mathbb{N}} \subset L^p(\Omega)$ converges weakly to $f \in L^p(\Omega)$ if for all $\phi \in L^q(\Omega)$ it holds that

$$\lim_{m\to\infty} \int_\Omega f_m \phi \, dx = \int_\Omega f \phi \, dx.$$

Definition A.8 (Weak* Convergence) *Let V be a Banach space, let V^* denote its dual space, let $\phi \in V^*$ and let $\{\phi_m\}_{m\in\mathbb{N}} \subset V^*$ be a sequence with the property that for all $f \in V$ it holds that*

$$\lim_{m\to\infty} \langle f, \phi_m\rangle = \langle f, \phi\rangle.$$

Then we say that the sequence $\{\phi_m\}_{m\in\mathbb{N}}$ converges in the weak sense to ϕ and we write $\phi_m \overset{*}{\rightharpoonup} \phi$.*

Remark A.9 *If the Banach space V is reflexive, i.e., $V = V^{**}$, the notions of weak convergence and weak* convergence coincide.*

The notion of weak and weak* convergence allows us to state the following important theorem from functional analysis:

Theorem A.10 (Banach-Alaoglu Theorem) *Let V be a normed vector space and let V^* denote its dual space. Then the unit closed ball $B^* = \{\phi \in V^* \colon \|\phi\|_{V^*} \leq 1\}$ in the dual space V^* is compact with respect to the weak* topology.*

Proof A detailed proof of this theorem can, e.g., be found in [Fol13, Theorem 5.18]. □

Corollary A.11 *Let V be a normed vector space, let V^* denote its dual space and let $\{\phi_m\}_{m\in\mathbb{N}} \subset V^*$ be a bounded sequence. Then there exists some $\phi \in V^*$ and a subsequence $\{\phi_{mk}\}_{k\in\mathbb{N}}$ such that $\phi_{m_k} \overset{*}{\rightharpoonup} \phi$.*

In other words, every bounded sequence in the dual space of a normed vector space contains a subsequence that converges in the weak sense.*

Corollary A.12 *Let V be a reflexive Banach space, let V^* denote its dual space and let $\{f_m\}_{m\in\mathbb{N}} \subset V$ be a bounded sequence. Then there exists some $f \in V$ and a subsequence $\{f_{mk}\}_{k\in\mathbb{N}}$ such that $f_{m_k} \rightharpoonup f$.*

In other words, every bounded sequence in a reflexive Banach space contains a subsequence that converges in the weak sense.

A partial converse to Corollary A.12 also exists:

Theorem A.13 (Boundedness of Weakly Convergent Sequences) *Let $d \geq 2$, let $1 \leq p < \infty$, let $\Omega \subseteq \mathbb{R}^d$, let $f \in L^p(\Omega)$ and let $\{f_m\}_{m\in\mathbb{N}}$ be a sequence of functions with the property that*

$$f_m \rightharpoonup f.$$

Then the sequence $\{f_m\}_{m\in\mathbb{N}}$ is bounded in $L^p(\Omega)$ and it holds that

$$\|f\|_{L^p(\Omega)} \leq \liminf_{m\to\infty} \|f_m\|_{L^p(\Omega)}.$$

Furthermore, if $1 < p < \infty$, $f_m \rightharpoonup f$ in $L^p(\Omega)$ and

$$\|f\|_{L^p(\Omega)} = \lim_{m\to\infty} \|f_m\|_{L^p(\Omega)},$$

then it holds that

$$\lim_{m\to\infty}\Big(\int_\Omega |f_m(x)-f(x)|^p dx\Big)^{\frac{1}{p}} = 0.$$

Thus, the sequence $\{f_m\}_{m\in\mathbb{N}}$ *converges in the strong* L^p *sense to* f*.*

Proof This theorem is, for instance, stated in [Eva90, Theorem 1.1.1]. The proof follows from an application of the uniform boundedness principle.

As the following example shows, weak convergence is a weaker notion than strong convergence. In particular, let $f_m \rightharpoonup f$ in some $L^p(\mathbb{R})$ space and let $g\colon \mathbb{R}\to\mathbb{R}$ be a general non-linear function. Then it does not necessarily hold that $g(f_m)\to g(f)$ as $m\to\infty$.

Example A.14 *Let* $a>0$*, let* $\{f_m\}_{m\in\mathbb{N}}\colon [0,1]\to\mathbb{R}$ *be a sequence of functions such that for all* $x\in[0,1]$ *and for all* $m\in\mathbb{N}$ *it holds that*

$$f_m(x) = \begin{cases} a & \text{for } x\in \big[j2^{-m},(j+1)2^{-m}\big) \quad \text{with } j=0,2,4,\dots,2^m-2,\\ -a & \text{for } x\in \big[j2^{-m},(j+1)2^{-m}\big) \quad \text{with } j=1,3,5,\dots,2^m-1,\end{cases}$$

and let $g\colon\mathbb{R}\to\mathbb{R}, g(x)=x^2$*.*

Then, clearly $\lim_{m\to\infty} g(f_m) = f_m^2 = a^2$*. On the other hand, the sequence* $\{f_m\}_{m\in\mathbb{N}}$ *is bounded in the space* $L^p\big([0,1]\big)$ *for every* $p\in[1,\infty)$*. It holds that* $f_m \rightharpoonup f\equiv 0$ *in* $L^p\big([0,1]\big)$ *for every* $p\in[1,\infty)$*.*

The preceding example indicates that we require some additional compactness condition in order to guarantee convergence in the case of non-linear functions.

Theorem A.15 (Arzelà-Ascoli Theorem) *Let* X *be a compact Hausdorff space, let* (Y,d) *be a complete metric space, let* $C(X,Y)$ *denote the set of continuous functions from* X *to* Y *and let* $\mathscr{F}\subseteq C(X,Y)$ *be a set with the property that*

- $\mathscr{F}$ *is uniformly, totally bounded, i.e., there exists a totally bounded set* $Z\subset Y$ *such that for all* $f\in\mathscr{F}$*,* $f(X)\subset Z$*, and*
- $\mathscr{F}$ *is equicontinuous in the sense that for all* $x\in X$ *and for all* $\epsilon>0$ *there exists a neighbourhood* U *of* x *such that for all* $\tilde{x}\in U$ *and for all* $f\in\mathscr{F}$ *it holds that* $d\big(f(x),f(\tilde{x})\big)<\epsilon$*.*

Then, $\bar{\mathscr{F}}$ is a compact subset of $C(X,Y)$ with respect to the uniform topology.

Proof A detailed proof can, for instance, be obtained by slightly modifying the proof stated in [Fol13, Theorem 4.44]. □

The Gagliardo-Nirenberg-Sobolev Inequality A.1 implies that the Sobolev space $W^{1,p}(\Omega)$ is embedded in the space $L^{p^*}(\Omega)$ for $1 \leq p < n, p^* = \frac{pd}{d-p}$. It turns out that for $1 \leq q < p^*$, this embedding is in fact a *compact embedding*.

Theorem A.16 (Rellich-Kondrachov Compactness Theorem) *Let $d \in \mathbb{N}$, let $p \in [1,d)$, let $p^* = \frac{pd}{d-p}$ and let $\Omega \subset \mathbb{R}^d$ be an open, bounded set with continuously differentiable boundary $\partial\Omega$. Then, for all $q \in [1,p^*)$ it holds that*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega).$$

In particular, since $p^* > p$ and $p^* \to \infty$ as $p \to d$, for all $p \in [1,\infty]$ it holds that

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega).$$

Proof A detailed proof of this theorem can, for example, be found in [Eva10, Section 5.7]. □

Theorem A.17 (Aubin-Lions Lemma) *Let $X \subset Y \subset Z$ be separable Banach spaces such that the embedding $X \subset Y$ is compact and the embedding $Y \subset Z$ is continuous, let $T > 0$, let $1 < p, q < \infty$ and let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions such that*

$$\{u_n\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^p\big(0,T;X\big) \tag{A1}$$

$$\{u_n'\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } L^q\big(0,T;Z\big), \tag{A2}$$

where $u_n' = \partial_t u_n$ denotes the derivative with respect to $t \in [0,T]$. Then the sequence $\{u_n\}_{n\in\mathbb{N}}$ is strongly pre-compact in $L^p\big(0,T;Y\big)$.

Consider the setting of Theorem A.17. The Aubin-Lions lemma then implies in particular that the sequence $\{u_m\}_{m\in\mathbb{N}}$ has a strongly convergent subsequence in $L^p\big(0,T;Y\big)$.

Proof Our proof requires use of the following lemma:

Lemma A.18 *Let $X \subset Y \subset Z$ be separable Banach spaces such that the embedding $X \subset Y$ is compact and the embedding $Y \subset Z$ is continuous. Then for all $\delta > 0$ there exists some constant $C_\delta > 0$ such that for all $x \in X$ it holds that*

$$\|x\|_Y \leq \delta\|x\|_X + C_\delta\|x\|_Z.$$

Proof The proof, which is based arguing by contradiction, is left as an exercise. ☐

Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence that satisfies the hypotheses (A1). It follows from the Banach-Alaoglu theorem A.10 that there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}} \subset L^p\big(0,T;X\big)$ and $u \in L^p\big(0,T;X\big)$ such that $u_{n_k} \overset{k\to\infty}{\rightharpoonup} u$ in $L^p\big(0,T;X\big)$. Throughout the remainder of this proof, we will use a simple relabelling to denote the subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ as $\{u_n\}_{n\in\mathbb{N}}$.

Without loss of generality we can prove that the sequence $\{u_n - u\}_{n\in\mathbb{N}} = \{v_n\}_{n\in\mathbb{N}}$ converges to 0 in the strong $L^p\big(0,T;Y\big)$ sense.

Note that Morrey's inequality A.5 implies that the set $\mathcal{Y}$ given by

$$\mathcal{Y} = \big\{y \in L^p\big(0,T;X\big) : \partial_t y \in L^q\big(0,T;Z\big)\big\},$$

is continuously embedded in the space $C\big([0,T];Z\big)$. Thus, there exists a constant C such that for all $n \in \mathbb{N}$ and for all $t \in [0,T]$ it holds that

$$\|v_n(t)\|_Z \leq C.$$

By the Lebesgue dominated convergence theorem, this implies that it is sufficient to show that for a.e. $t \in [0,T]$ the sequence $v_n(t) \to 0$ in Z as $n \to \infty$.

Once again without loss of generality, we let $t = 0$. Then for all $n \in \mathbb{N}$ it holds that

$$\begin{aligned} v_n(0) &= v_n(t) - \int_0^t v_n'(\tau)\,d\tau \\ &= \frac{1}{s}\Big(\int_0^s v_n(t)dt - \int_0^s \int_0^t v_n'(\tau)\,d\tau dt\Big) \\ &= a_n + b_n. \end{aligned}$$

Moreover, for all $n \in \mathbb{N}$ we can rewrite b_n as

$$b_n = -\frac{1}{s}\int_0^s (s-t)v_n'(t)dt.$$

Thus, given any $\epsilon > 0$, we can choose $s > 0$ such that for all $n \in \mathbb{N}$ it holds that

$$\|b_n\|_Z \leq \int_0^s \|v_n'(t)\|_Z \, dt \leq \frac{\epsilon}{2}$$

Since $v_n \rightharpoonup 0$ weakly in X as $n \to \infty$, $a_n \to 0$ strongly in Z thanks to the compact embedding $X \subset\subset Y$ and the continuous embedding $Y \subset Z$. Furthermore, for this fixed s, the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges weakly to 0 as $n \to \infty$. Due to the compact embedding of $X \subseteq Y$ and the continuous embedding $Y \subseteq Z$, it follows that the sequence $\{a_n\}_{n\in\mathbb{N}}$ also converges strongly to 0 in Z as $n \to \infty$. Therefore, for sufficiently large n it holds that $\|a_n\|_Z \leq \frac{\epsilon}{2}$. We conclude that for $t \in [0, T]$ it holds that

$$\lim_{n\to\infty} v_n(t) = 0 \qquad \text{strongly in } Z.$$

Hence $v_n \to 0$ in $C([0, T]; Z)$. Since $[0, T]$ is bounded also $v_n \to 0$ in $L^p(0, T; Z)$.

We now claim that this is sufficient to show that $\{v_n\}_{n\in\mathbb{N}} \to 0$ in the strong $L^p\big(0, T; Y\big)$ sense. Indeed, Lemma A.18 implies that for all $\delta > 0$ there exists some constant $C_\delta > 0$ such that for all $n \in \mathbb{N}$ it holds that

$$\|v_n\|_{L^p(0,T;Y)} \leq \delta\|v_n\|_{L^p(0,T;X)} + C_\delta\|v_n\|_{L^p(0,T;Z)}.$$

Since $\{v_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^p\big((0, T; X)\big)$, it follows that for all $n \in \mathbb{N}$ it holds that

$$\|v_n\|_{L^p(0,T;Y)} \leq C\delta + C_\delta\|v_n\|_{L^p(0,T;Z)}.$$

Passing to the limit $n \to \infty$ we obtain

$$\limsup_{n\to\infty} \|v_n\|_{L^p(0,T;Y)} \leq C\delta. \qquad \square$$

Since δ can be chosen arbitrarily small, this proves the claim. Hence $v_n \to 0$ in $L^p(0, T; Y)$.

A.3 Miscellaneous Results

The following proposition will be used in Chapter 2.

Proposition A.19 *Let $d \in \{2,3\}$, let $\Omega \subseteq \mathbb{R}^d$ be an open, bounded Lipschitz domain, and let $f \in H^{-1}(\Omega;\mathbb{R}^d)$ be a function with the property that for all $v \in \mathcal{V}$ it holds that $\langle f, v\rangle = 0$. Then, there exists some scalar field $p \in L^2(\Omega)$ with the property that*

$$f = \nabla p.$$

Proof This result is a consequence of Proposition 1.1 and Proposition 1.2 in [Tem01, Chapter 1]. See also Remark 1.4 [Tem01, Chapter 1]. □

The following theorem is one of the fundamental tools for proving existence and uniqueness of solutions to the variational formulation of certain types of elliptic partial differential equations.

Theorem A.20 (Lax-Milgram Lemma) *Let H be a real Hilbert space, let $a\colon H\times H \to \mathbb{R}$ be a continuous and coercive bilinear form and let $\ell\colon H \to \mathbb{R}$ be a bounded linear functional. Then there exists a unique element $u \in H$ with the property that for all $v \in H$ it holds that*

$$a(u,v) = \ell(v).$$

Proof The proof relies on the Riesz Representation theorem for Hilbert spaces. A detailed proof can, e.g., be found in [Eva10, Section 6.2.1] □

Theorem A.21 *Let $d \in \mathbb{N}$, let $q > d$, let $\Omega \subset \mathbb{R}^d$ satisfy an exterior cone condition at each point of the boundary $\partial\Omega$, let $\phi \in C^0(\partial\Omega)$, let $g \in L^{q/2}(\Omega), f^i \in L^q(\Omega), (i = 1,\dots,d)$ and let $a^{ij}, b^i, c^i, e\,(i,j = 1,\dots,d)$ be measurable functions with the property that there exists $\lambda > 0$ such that for all $\xi \in \mathbb{R}^d$ it holds that*

$$\sum_{i,j=1}^{d} a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2,$$

with the property that there exists $\Lambda, \nu \geq 0$ such that for all $x \in \Omega$

$$\sum_{i,j=1}^{d} |a^{ij}(x)|^2 \leq \Lambda^2, \qquad \lambda^{-2}\sum_{i=1}^{d}\big(|b^i(x)|^2 + |c^i(x)|^2\big) + \lambda^{-1}|e(x)| \leq \nu^2,$$

and with the property that for all $i \in \{1,\dots,d\}$ and for all non-negative test functions $v \in C_0^1(\Omega)$ it holds that

$$\int_\Omega \big(e(x)v(x) - b^i(x)\partial_i v(x)\big)\,dx \leq 0.$$

Then, the generalised Dirichlet problem given by

$$Lu := \sum_{i=1}^{d} \left(\partial_i \big(\sum_{j=1}^{d} a^{ij}(x) \partial_j u + b^i(x) u \big) + c^i(x) \partial_i u \right) + e(x) u = g + \sum_{i=1}^{d} \partial_i f^i \quad \text{on } \Omega,$$

$$u = \phi \qquad \text{on } \partial\Omega,$$

is uniquely solvable and moreover, the solution $u \in H^1_{loc}(\Omega) \cap C^0(\bar{\Omega})$.

Proof This theorem and its proof can be found in [GT15, Chapter 8, Theorem 8.30] □

B Basic Notation

Throughout these lecture notes, the following basic notation will be used.

Notation B.1 (Partial derivatives) *For a function* $f : [0,T]\times\Omega \to \mathbb{R}^m$, $\Omega \subset \mathbb{R}^d$, $m \in \mathbb{N}$, *we may use the notation*

$$\frac{\partial f}{\partial t} = \partial_t f$$

to denote the partial derivatives with respect to time and

$$\frac{\partial f}{\partial x_i} = \partial_{x_i} f = \partial_i f, \quad i = 1, \dots, d,$$

to denote the partial derivatives with respect to x_i, $i = 1, \dots, d$.

Definition B.2 (Divergence of a Vector Field) *Let* $d \in \mathbb{N}$, *let* $\Omega \subseteq \mathbb{R}^d$, *let* $f := (f_1, f_2, \dots f_d) \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ *be a smooth vector field. Then we denote by* $\operatorname{div} f \in \mathbb{R}$ *the mapping given by*

$$\operatorname{div} f = \sum_{i=1}^{d} \frac{\partial f_i}{\partial x_i},$$

and we call this mapping the divergence of f.

Definition B.3 (Divergence of a Tensor) *Let* $d \in \mathbb{N}$, *let* $\Omega \subseteq \mathbb{R}^d$, *let* $f := \{f_{i,j}\}_{i,j=1,\dots,d} \colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^{d\times d}$ *be a smooth tensor. Then we denote by* $\operatorname{div} f \in \mathbb{R}^d$ *the mapping given by*

$$\operatorname{div} f = \begin{bmatrix} \sum_{i=1}^{d} \frac{\partial f_{1,i}}{\partial x_i} \\ \sum_{i=1}^{d} \frac{\partial f_{2,i}}{\partial x_i} \\ \vdots \\ \sum_{i=1}^{d} \frac{\partial f_{d,i}}{\partial x_i} \end{bmatrix},$$

and we call this mapping the divergence of f.

We remark that we are using identical notation for the divergence of both vectors and tensors; the distinction will be clear based on the context.

Definition B.4 (Gradient of a Scalar Field) *Let $d \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^d$, let $f\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be a smooth scalar field. Then we denote by $\nabla f \in \mathbb{R}^d$ the mapping given by*

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_d} \end{bmatrix},$$

and we call this mapping the gradient of f.

Definition B.5 (Gradient of a Vector Field) *Let $d \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^d$, let $f := (f_1, f_2, \ldots f_d)\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ be a smooth vector field. Then we denote by $\nabla f \in \mathbb{R}^d$ the mapping given by*

$$\nabla f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_d}{\partial x_1} & \cdots & \frac{\partial f_d}{\partial x_d} \end{bmatrix},$$

and we call this mapping the gradient of f.

We remark once again that we are using identical notation for the gradient of both scalars and vectors; the distinction will be clear based on the context.

Definition B.6 (Laplacian of a Scalar Field) *Let $d \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^d$, let $f\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ be a smooth scalar field. Then we denote by Δf the mapping given by*

$$\Delta f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2},$$

and we call this mapping the Laplacian of f.

Definition B.7 (Laplacian of a Vector Field) *Let $d \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^d$, let $f := (f_1, f_2, \ldots f_d)\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ be a smooth vector field. Then we* *denote by Δf the mapping given by*

$$\Delta f = \begin{bmatrix} \sum_{i=1}^{d} \frac{\partial^2 f_1}{\partial x_i^2} \\ \sum_{i=1}^{d} \frac{\partial^2 f_2}{\partial x_i^2} \\ \vdots \\ \sum_{i=1}^{d} \frac{\partial^2 f_d}{\partial x_i^2} \end{bmatrix},$$

and we call this mapping the Laplacian of f.

Once more we are using identical notation for the Laplacian of both scalars and vectors; the distinction will be clear based on the context.

Definition B.8 (Product of Vector Gradients) *Let $d \in \mathbb{N}$, let $\Omega \subseteq \mathbb{R}^d$, let $f := (f_1, f_2, \dots f_d)\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ and $g := (g_1, g_2, \dots g_d)\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}^d$ be two smooth vector fields. Then we denote by $\nabla f : \nabla g$ the mapping given by*

$$\nabla f : \nabla g = \sum_{j=1}^{d} \sum_{i=1}^{d} \frac{\partial f_j}{\partial x_i} \frac{\partial g_j}{\partial x_i}.$$

If $f = g$, we may write $\nabla f : \nabla f = |\nabla f|^2$.

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