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ETH Zürich
D-MATH

Topics in Mathematical and Computational Fluid Dynamics

Problem Sheet 1

Problem 1.1 The trilinear form b

Consider the trilinear form defined by

$$\begin{aligned} b &: H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R} \\ b(u,v,w) &:= \int_\Omega [(u \cdot \nabla)v] \cdot w \, dx. \end{aligned} \tag{1.1.1}$$

1.1a) Show that b is continuous on $H_0^1(\Omega) \times H_0^1(\Omega) \times H_0^1(\Omega)$ for bounded domains $\Omega \subset \mathbb{R}^d$, $d \leq 4$.

Hint: Use the Gagliardo-Nirenberg interpolation inequality.

1.1b) Show that for u divergence free ($\operatorname{div} u = 0$), b is skew-symmetric in the second and third argument, that is, show

$$b(u,v,w) = -b(u,w,v),$$

for $u \in V := H_0^1(\Omega) \cap \{\operatorname{div} f = 0\}$, $v, w \in H_0^1(\Omega)$.

Problem 1.2 An auxilary lemma to prove the Aubin-Lions Lemma

In the proof of the Aubin-Lions lemma, the following lemma is useful:

Lemma 1.1. *Let X, Y, Z be three Banach spaces such that $X \subset Y \subset Z$ and the injection of X into Y is compact and the injection of Y into Z is continuous. Then, for any $\delta > 0$, there exists some constant C_δ depending on δ (and the spaces X, Y and Z) such that*

$$\|v\|_Y \leq \delta \|v\|_X + C_\delta \|v\|_Z, \quad \forall\, v \in X.$$

Prove this lemma.

Hint: Argue by contradiction. Assume that the contrary statement is true, i.e. there exists some $\delta > 0$ such that for any $c \in \mathbb{R}$,

$$\|v\|_Y \geq \delta \|v\|_X + c\|v\|_Z,$$

for at least one $v \in X$.

If you need more inspiration, recall or look up the proof of Poincaré's inequality (e.g. Evans: Partial Differential Equations, p. 290).

Problem 1.3 Continuous version of Aubin-Lions lemma

In the lecture, we saw the L^p-version of the Aubin-Lions lemma. One can also show a different version of the lemma, the so-called C^0-version which is the following:

Lemma 1.2. *Let X, Y, Z be three separable, reflexive Banach spaces such that $X \subset Y \subset Z$ and the injection of X into Y is compact and the injection of Y into Z is continuous. Let $T > 0$ and assume that the sequence of functions $\{u_n\}_{n\geq 1}$ satisfies for an $M > 0$,*

- $\|u_n(t)\|_X \leq M$ *for all* $t \in [0, T]$,
- $\{u_n\}_{n\geq 1}$ *is uniformly equicontinuous on* $[0, T]$ *with values in* Z.

Then the sequence $\{u_n\}_{n\geq 1}$ is precompact in $C([0, T]; Y)$.

The proof of this lemma is much shorter than the one for the L^p-version given the lemma from the previous exercise. Prove this lemma.

Hint: Use the lemma from the previous problem to show that $u_n \in C([0, T]; Y)$ and is uniformly equicontinuous. Then apply the Arzela-Ascoli theorem.

Problem 1.4 Weak formulation of the Stokes problem

In the lecture, we talked about the Stokes problem:

$$\begin{aligned} -\mu\Delta u + \nabla p &= f, \quad \text{in } \Omega \\ \operatorname{div} u &= 0, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega. \end{aligned}$$

We stated the following theorem which shows that the Stokes problem can be reformulated as a variational formulation:

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^d$ be open bounded and its boundary of class C^2. Then the following are equivalent:*

(i) u *is a weak solution of the Stokes equation, i.e. it satisfies*

$$\begin{aligned} &u \in V := H_0^1(\Omega) \cap \{\operatorname{div} f = 0\}, \\ &a(u, v) = (f, v), \quad \forall v \in \mathcal{V} := C_c^\infty(\Omega) \cap \{\operatorname{div} f = 0\}. \end{aligned} \tag{1.4.1}$$

(ii) $u \in H_0^1(\Omega)$ *satisfies: There exists* $p \in L^2(\Omega)$ *such that*

$$\begin{aligned} -\mu\Delta u + \nabla p &= f, \quad \textit{in the sense of distributions}, \\ \operatorname{div} u &= 0, \quad \textit{in the sense of distributions}, \\ u &= 0, \quad \textit{in the sense of traces.} \end{aligned}$$

(iii) $u \in V$ *achieves the minimum of* $\Phi(v) = a(v, v) - 2(f, v)$ *on* V.

where the bilinear forms $(\cdot,\cdot)$ and $a(\cdot,\cdot)$ are defined by

$$(f,v) := \int_\Omega f \cdot v \, dx$$
$$a(u,v) := \mu \int_\Omega \nabla u \cdot \nabla v \, dx$$

Prove this theorem.

Hint: Try to show (i) $\iff$ (ii) and (i) $\iff$ (iii).

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To be submitted by March 24. Submit solved assignements to F. Leonardi or before/after class if you decide to do so.