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Sheet 1

1. a) Let A be a fixed subset of a set X. Determine the σ-algebra of subsets of X generated by $\{A\}$.

b) Let X be an uncountable set; let

$$\mathcal{S} = \{E \subset X : E \text{ or } E^c \text{ is at most countable}\}.$$

Show that $\mathcal{S}$ is the σ-algebra generated by the one-point subsets of X.

2. Let (X, μ) be a measure space and E a measurable subset of X. Show that for every $A \subset X$ the following holds:

$$\mu(E \cap A) + \mu(E \cup A) = \mu(E) + \mu(A).$$

3. Let μ be a measure on a set X, and let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of X satisfying

$$\sum_{n=1}^{\infty} \mu(A_n) < \infty.$$

Consider the set

$$E = \{x \in X : x \text{ belongs to } A_n \text{ for infinitely many } n\};$$

show that $\mu(E) = 0$.

4. Let $A_1 \supset A_2 ... \supset A_k \supset A_{k+1} ...$ be a sequence of measurable subsets of a space X endowed with a measure μ. If $\mu(A_1) < \infty$, we know that

$$\lim_{k \to \infty} \mu(A_k) = \mu\left(\cap_{k=1}^{\infty} A_k\right)$$

(see Evans-Gariepy page 2, Theorem 1.(iv)). Show, by means of a counterexample, that this relation no longer holds if we drop the assumption $\mu(A_1) < \infty$.

5. Let $\mathbb{N}$ be the set of natural numbers. Take the function $\mu : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ defined as follows: $\mu(A) = 0$ if A is a finite subset of $\mathbb{N}$, $\mu(A) = \infty$ if A is infinite. Show that μ is not a measure on $\mathbb{N}$.