Sheet 2

1. Show that the Borel sets of $\mathbb{R}^n$ are exactly the members of the σ-algebra generated by the compact sets.

2. Let λ denote the Lebesgue measure on $\mathbb{R}^n$. Show that

a) if $E \subset [0,1]^n$ satisfies $\lambda(E) = 1$, then E is dense in $[0,1]^n$;

b) if $E \subset \mathbb{R}^n$ satisfies $\lambda(E) = 0$, then E has empty interior.

3. Denote by λ the Lebesgue measure on $\mathbb{R}$. Let $E \subset [0,1]$ be a Lebesgue measurable set of strictly positive measure, $\lambda(E) > 0$. Show that for any $0 \leq \delta \leq \lambda(E)$ there exist a measurable subset of E having measure δ.

Hint: Observe the function which associates to $t \in [0,1]$ the measure of $[0,t] \cap E$. Is it continuous?

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function, with Lipschitz constant K. Show that, if $E \subset \mathbb{R}$ has zero Lebesgue measure, then $f(E)$ is also a null set.

Hint: Observe how f behaves on an interval J.