Sheet 3

1. Show that the following system of elementary objects

$$\mathcal{A} := \{A \subset \mathbb{R}^n \mid A \text{ is the union of finitely many disjoint intervals}\}$$

is an algebra. [1]

Remark: An interval I in $\mathbb{R}^n$ is of the form $I = I_1 \times \ldots \times I_n$, $1 \leq k \leq n$, for I_k intervals in $\mathbb{R}$.

2. Prove: A subset $\Omega \subset \mathbb{R}^n$ has Lebesgue measure zero if and only if there exists a sequence $(I_k)_{k\in\mathbb{N}}$ of intervals with the following properties:

1. $\sum_{k=1}^{\infty} \mathrm{Vol}(I_k) < \infty$.
2. Every point of Ω lies in infinitely many intervals I_k.

Remark: For $I = I_1 \times \ldots \times I_n$ an interval in $\mathbb{R}^n$, the volume is defined by $\mathrm{Vol}(I) = \Pi_{k=1}^{n} \mathrm{Vol}(I_k)$, for $\mathrm{Vol}(I_k)$ the length of I_k (in $\mathbb{R}$).

3. Let X be an uncountable set and

$$\mathcal{B} = \{\, E \subseteq X \ : \ E \text{ countable or } E^c = X \setminus E \text{ countable} \,\} \ .$$

be a σ-algebra (see sheet 1, 1.b). Show that $\mu\colon \mathcal{B} \to [0,1]$ with

$$\mu(E) = \begin{cases} 0 & \text{if } E \text{ countable} \\ 1 & \text{else} \end{cases}$$

is a premeasure on $\mathcal{B}$.

[1] See Remark 1.3.1 in the script *Analysis 3*, Struwe

4. a) The counting measure $\mu : \mathcal{P}(X) \to [0, \infty]$ is defined by

$$\mu(A) := \begin{cases} \#A & \text{if A is finite} \\ \infty & \text{else} \end{cases}$$

Prove: μ is a measure on $\mathbb{R}^n$.

b) Let $x \in X$. The Dirac measure $\delta_x : \mathcal{P}(X) \to [0, \infty]$ is defined by

$$\delta_x(A) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Prove: δ_x is a measure on $\mathbb{R}^n$.

5. a) Give an example of a sequence $\{E_k\}_{k=1}^{\infty}$ of pairwise disjoint subsets of some measure space (X, μ) such that

$$\mu\left(\cup_{k=1}^{\infty} E_k\right) < \sum_{k=1}^{\infty} \mu(E_k).$$

b) For any $\epsilon > 0$ give an example of a compact $K_\epsilon \subset [0, 1] \subset \mathbb{R}$ with empty interior and such that its Lebesgue measure is at least $1 - \epsilon$.