Sheet 4

1. Prove that the Lebesgue measure is invariant under translations and rotations, i.e. under all motions of the form

$$\Phi : \mathbb{R}^n \to \mathbb{R}^n, \qquad \Phi(x) = x_0 + Rx$$

for $x_0 \in \mathbb{R}^n$ and $R \in O(n)$.

2. Show that every countable subset of $\mathbb{R}$ is a Borel set and has Lebesgue measure zero.

3. The goal of this exercise is to show that the Cantor triadic set C is uncountable. For that, recall quickly the construction of C: Every $x \in [0,1]$ can be expanded in base 3, i.e. can be written as $x = \sum_{i=1}^{\infty} d_i(x)\frac{1}{3^i}$ for $d_i(x) \in \{0,1,2\}$. The set C is then defined as the set of those $x \in [0,1]$ which do not have any digit 1 in their 3-expansion, i.e.

$$C := \{x \in [0,1] | d_i(x) \in \{0,2\} \forall i\}.$$

Now, the Cantor-Lebesgue function F is defined by

$$F : C \to [0,1], \qquad F\left(\sum_{i=1}^{\infty} a_i \frac{1}{3^i}\right) = \sum_{i=1}^{\infty} a_i \frac{1}{2^{i+1}}$$

a) Show that $F(0) = 0$ and $F(1) = 1$.

b) Show that F is well-defined and continuous on C.

c) Show that F is surjective.

d) Conclude that C is uncountable.

4. Let $A \subset \mathbb{R}^n$ and $0 \leq s < t \leq \infty$. Show that:

(i) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.

(ii) If $\mathcal{H}^t(A) > 0$, then $\mathcal{H}^s(A) = +\infty$.

Observe that this implies that there is a unique value $d \in \mathbb{R}$ such that $\mathcal{H}^s(A) = 0$ for $s > d$ and $\mathcal{H}^s(A) = +\infty$ for $s < d$. The value d is called the Hausdorff dimension of A.