Sheet 10

1. a) For which $s > 0$ is

$$\int_a^b \frac{1}{x^s} dx < \infty ,$$

for $(a, b) = (0, 1), (1, \infty), (0, \infty)$?

b) The Gamma-function is defined by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx ,$$

for $s \in \mathbb{R}$, $s > 0$. Show that Γ is everywhere differentiable and calculate its differentials as integrals.

2. Let $\mu(\Omega) < \infty$ and $f, f_k : \Omega \to \overline{\mathbb{R}}$ μ-summable.

a) Show that Vitali's Theorem implies Lebesgue's Theorem about dominated convergence.

b) Let $\Omega = [0, 1]$. Give an example in which Vitali's Theorem can be applied but there does not exists a dominating function $g \in L^1([0, 1])$.
Hint: Look at $f_k = \chi_{[\frac{k-2^n}{2^n}, \frac{k+1-2^n}{2^n}[}$

3. **(Generalized Hölder-inequality)** Let $1 \leq p_1, \ldots, p_k \leq \infty$ be given such that $\frac{1}{r} = \sum_{i=1}^k \frac{1}{p_i} \leq 1$. Show that for $f_i \in L^{p_i}(\Omega, \mu)$ it holds $\prod_{i=1}^k f_i \in L^r(\Omega, \mu)$ and

$$\left\| \prod_{i=1}^k f_i \right\|_{L^r} \leq \prod_{i=1}^k \|f_i\|_{L^{p_i}} .$$

4. Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that $f_n \xrightarrow{n\to\infty} f$ in $L^p(\Omega, \mu)$ implies

$$\int_\Omega f_n \, g \, d\mu \xrightarrow{n\to\infty} \int_\Omega f \, g \, d\mu$$

for all $g \in L^q(\Omega)$.