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Sheet 11

1. Let (X,μ) and (Y,ν) be two measure spaces. Suppose that the functions $g : X \to \mathbb{R}$ and $h : Y \to \mathbb{R}$ are respectively μ-measurable and ν-measurable. Consider the function

$$f : X \times Y \to \mathbb{R} \qquad \text{with} \qquad f(x,y) := g(x)\,h(y) .$$

Show that f is $(\mu \times \nu)$-measurable and establish the identity

$$\int_{X\times Y} f \, d(\mu\times\nu) = \int_X g\, d\mu \int_Y h\, d\nu .$$

2. Let $0 < a < b$. Consider the function x^y on the domain $\{0 \le x \le 1\} \times \{a \le y \le b\}$. By applying Fubini's Theorem to the integral

$$\int_{[0,1]\times[a,b]} x^y dxdy$$

show that

$$\int_0^1 \frac{x^b - x^a}{\ln x} dx = \ln\left[\frac{1+b}{1+a}\right].$$

3. Let $X = \mathbb{R}^k$, $Y = \mathbb{R}^l$ be endowed with the measures $\mu = \mathcal{L}^k$, $\nu = \mathcal{L}^l$, respectively the k- and l-dimensional Lebesgue measure.

a) Show that the product measure $\mu \times \nu$ on $X \times Y = \mathbb{R}^{k+l}$ is given by

$$(\mu\times\nu)(S) = \inf\{(\mu\times\nu)(G) : G \subset \mathbb{R}^{k+l} \text{ is open }, S \subset G\}.$$

b) Show that $\mu \times \nu$ is just the $(k+l)$-dimensional Lebesgue measure $\mathcal{L}^{k+l}$ on $\mathbb{R}^{k+l}$.

4. a) Show that any $f \in \bigcap_{p\in\mathbb{N}} L^p(\Omega,\mu)$ with $\sup_{p\in\mathbb{N}} \|f\|_{L^p} < \infty$ lies in $L^\infty(\Omega,\mu)$ as well.

Hint: Tchebychev' inequality.

b) Show that if $\mu(\Omega) < \infty$, then for any f as in **a)**, we have that $\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}$.

c) Find an $f \in \bigcap_{p\in\mathbb{N}} L^p(\Omega,\mu)$ with $f \notin L^\infty(\Omega,\mu)$, such that the property from **a)** is not satisfied without the assumption $\sup_{p\in\mathbb{N}} \|f\|_{L^p} < \infty$.