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Sheet 12

1. Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, where p and q belong to $(1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Show that the convolution product $f * g$ is a continuous function on $\mathbb{R}$; then prove that $f * g$ goes to 0 at infinity, i.e. that for every $\epsilon > 0$ there exists $\delta > 0$ with

$$\sup_{|x|>\delta} \left|(f * g)(x)\right| \ < \ \epsilon \, .$$

Hint: Approximate f and g by functions in C_0^0.

2. Let $f \in L^p(\mathbb{R}, \lambda)$, where λ is the Lebesgue measure. By means of Fubini's Theorem, show that the following equality holds:

$$\int_{\mathbb{R}} |f(x)|^p dx = p \int_0^\infty y^{p-1} \lambda(\{x \in \mathbb{R} : |f(x)| \geq y\}) dy.$$

Hint: $|f(x)|^p = \int_0^{|f(x)|} p y^{p-1} dy$.

Remark: Compare with series 9, ex.1. In that case there was an underlying Fubini-type argument in the proof. This time we can use Fubini's theorem and get a straightforward proof.

3. Define $f : [0,1]^2 \to \mathbb{R}$ by

$$f(x,y) := \begin{cases} y^{-2}, & \text{if } 0 < x < y < 1, \\ -x^{-2}, & \text{if } 0 < y < x < 1, \\ 0, & \text{else.} \end{cases}$$

Is this function summable w.r.t the Lebesgue measure?

4. Marcinkiewicz Theorem

a) Let $F \subset [0,1]$ be a closed set and $\delta_F(z) = \inf\{|y-z|; y \in F\}$. Using Fubini's Theorem, show

$$\int_0^1 \frac{\delta_F(z)^\lambda}{|x-z|^{1+\lambda}}\, dz < \infty$$

for all $\lambda > 0$ and almost every $x \in F$.

Hint: Begin by showing that $\delta_F(x) = 0$ is equivalent to $x \in F$, if F is closed.

b) Let $F \subset \mathbb{R}$ be closed and $f \geq 0$ summable on the complement of F. Show: For every $\lambda > 0$ the function

$$x \mapsto \int_{\mathbb{R}} \frac{\delta_F(z)^\lambda}{|x-z|^{1+\lambda}}\, f(z)\, dz$$

is summable on F.