Sheet 13

1. Let $\mathbb{R}^n$ be endowed with the Lebesgue measure $\mathcal{L}^n$ and consider a measurable subset $E \subset \mathbb{R}^n$. Denote by $B_r(x)$ the ball of radius r centered at x. Show that, for $\mathcal{L}^n$-a.e. $x \in E$ it holds

$$lim_{r\to 0} \frac{\mathcal{L}^n\left(E \cap B_r(x)\right)}{\mathcal{L}^n\left(B_r(x)\right)} = 1.$$

2. Let μ be a finite Borel measure on $[1, \infty)$ satisfying

(i) $\mu \ll \lambda$ with continuous Radon-Nikodým derivative $d\mu/d\lambda = f$.

(ii) $\mu(B) = \alpha^2 \mu(\alpha B)$ for each $\alpha \geq 1$ and each Borel subset $B \subset [1, \infty)$.

Prove that there exists some nonnegative constant M such that

$$f := \frac{M}{x^3} \qquad \forall\ x \geq 1\ .$$

3. Let $g(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$ for $x \in \mathbb{R}^2 \setminus \{0\}$. Show that $u = f * g$ solves the Laplace-Equation

$$-\Delta u = f$$

for $f \in C_c^\infty(\mathbb{R}^2)$.

4.
a) Is the Hardy-Littlewood maximal function f^* of $f = \chi_{[0,1]}$ summable?
b) Find the smallest C with $\mathcal{L}_1(\{f^* > \alpha\}) \leq \frac{C}{\alpha}$ for all $\alpha > 0$, with $f = \chi_{[0,1]}$.

5.
a) Show: For $f \in L^\infty(\mathbb{R}^n)$ is $f^* \in L^\infty(\mathbb{R}^n)$ and $\|f^*\|_{L^\infty} \leq \|f\|_{L^\infty}$.
b) Show that $(f + g)^* \leq f^* + g^*$ for all $0 \leq f, g \in L^1_{\text{loc}}(\mathbb{R}^n)$.
c) Find f and g with $(f + g)^*(x) < f^*(x) + g^*(x)$ on a set of positive measure.