Sheet 5

1. Let $\gamma : [a, b] \to X$ be a continuous injective curve in the metric space (X, d). We define the arc length of γ as

$$L(\gamma) = \sup \sum_{i=1}^{N} d(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all $N \in \mathbb{N}$ and all $a \leq t_0 \leq \ldots \leq t_N \leq b$. Show: $\mathcal{H}^1(\mathrm{Im}(\gamma)) = \frac{1}{2}L(\gamma)$.

2. a) Let I be the interval $[0, 1]$ in $\mathbb{R}$. Prove that $\mathcal{H}^1(I) = \frac{1}{2}$.

b) Perform inductively the following construction: start with a closed equilateral triangular region E_0 of side 1. Let E_1 be made of the three closed equilateral triangular regions T_1^l ($l = 1, 2, 3$) of side $\frac{1}{3}$ which are inside E_0 and are located in the corners of E_0. In other words we are removing an open hexagon of side $\frac{1}{3}$ from the middle of E_0.

At each step of the construction, we start from a set E_{j-1} which is made of the 3^{j-1} closed triangular regions T_{j-1}^l (for $l = 1, ..., 3^{j-1}$). On each T_{j-1}^l we perform the analogous construction: we take the three closed equilateral triangular regions which are located at the corners of T_{j-1}^l and have sides of length $\frac{1}{3}$ of the sides of T_{j-1}^l; performing this in each T_{j-1}^l we get 3^j triangles T_j^l, whose union we denote by E_j.

Set $E = \cap_{j=1}^{\infty} E_j$.

Show that the Hausdorff dimension of E is 1.

Hint: for each positive δ of a sequence going to 0, find a suitable cover to show that $\mathcal{H}^1(E)$ is bounded from above by a positive number. To get a positive lower bound, observe the projection of E onto one of the sides of E_0.

3. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz with constant L. Let $A \subset \mathbb{R}^n$ and $0 \leq s < +\infty$. Show that

$$H^s(f(A)) \leq L^s H^s(A).$$

Bitte wenden!

4. Consider a measure μ on $\mathbb{R}^n$ and a μ-measurable $X \subset \mathbb{R}^n$. Let f be an $\mathbb{R}$-valued function on X. f is called μ-measurable if for every open subset U of $\mathbb{R}$, the set $f^{-1}(U)$ is μ-measurable. Moreover, the sets $f^{-1}(\{-\infty\})$ and $f^{-1}(\{\infty\})$ need to be μ-measurable.

Now take a dense subset A of $\mathbb{R}$. Show that f is measurable if and only if the set $\{x \in X : f(x) \geq a\}$ is measurable for each $a \in A$.

5. Let E be the collection of all numbers in $[0, 1]$ whose decimal expansion has no sevens appearing.

Recall that some decimals have two possible expansions. We are taking the convention that no expansion should be identically zero from some digit onward; for example $\frac{27}{100}$ should be written as $0, 269999....$ and not as $0, 27$.

Prove that E is a Lebesgue-measurable set and determine its Lebesgue measure.