Sheet 7

1. Let $f_k\colon \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-measurable functions ($k \in \mathbb{N}$). Let

$$\mathcal{L}^n(\{x\,;\,|f_k(x) - f_{k+1}(x)| > 2^{-k}\}) < 2^{-k}$$

for all $k \in \mathbb{N}$. Show: The limit $\lim\limits_{k\to\infty} f_k(x)$ exists almost everywhere.

2. Let f be a finite, μ-measurable function, and $(f_k)_{k\in\mathbb{N}}$ a sequence of μ-measurable functions with the following property: Every $(f_{k_j})_{j\in\mathbb{N}}$ contains a subsequence, which converges to f in measure μ.

a) Show that the whole sequence $(f_k)_{k\in\mathbb{N}}$ converges to f in measure μ.

b) Show that the analogous statement from a) is not true, if we assume only pointwise convergence μ-almost everywhere.

3. Counter example to $\delta = 0$ in Egoroff's Theorem: Find an example of a sequence of $\mathcal{L}^1$-measurable functions $f_k : [0,1] \to \overline{\mathbb{R}}$, which converges almost everywhere pointwise to the function f, but for every compact $F \subset [0,1]$ with $\mathcal{L}^1(F) = \mathcal{L}^1([0,1])$ the convergence on F is not uniformly.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue-measurable function with

$$f(x+y) = f(x) + f(y)\ .$$

a) Show that f is continuous.
Hint: Use Lusin's Theorem to show that f is continuous at $x = 0$.

b) Show that

$$f(x) = x \cdot f(1)\ .$$

5. Take a Radon measure μ on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a μ-measurable subset. Consider a function $f : \Omega \to \mathbb{R}$ which is 0 μ-a.e. Show that f is summable with $\int_\Omega f = 0$.