Sheet 8

1. Proof the following Theorem: Let $f : \Omega \to \overline{\mathbb{R}}$ be a μ-summable function and

$$|\int_\Omega f \, d\mu| = \int_\Omega |f| \, d\mu \, .$$

Then either $f \geq 0$ or $f \leq 0$ almost everywhere on Ω.

2. Let $f :]0,1[\to \mathbb{R}$ be summable. Show that $x^k f$ ist summable as well for all $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \int_0^1 x^k f(x) \, dx = 0 \, .$$

3. Find an example of a continuous, bounded function $f : [0,\infty[\to \mathbb{R}$ with the asymptotic property $\lim_{x\to\infty} f(x) = 0$, such that

$$\int_0^\infty |f(x)|^p dx = \infty \, ,$$

for all $p > 0$.

4. a) Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of functions on a measurable set $\Omega \subset \mathbb{R}^n$. Show that the series $\sum_{k=1}^\infty f_k(x)$ converges almost everywhere, if

$$\sum_{k=1}^\infty \int_\Omega |f_k| dx < \infty \, .$$

b) Let $\{r_k\}$ be an ordering of $\mathbb{Q} \cap [0,1]$; $(a_k)_{k\in\mathbb{N}} \subset \mathbb{R}$, such that $\sum_{k=1}^\infty a_k$ is absolute convergent. Show that then $\sum_{k=1}^\infty a_k |x - r_k|^{-1/2}$ is absolute convergent for almost every $x \in [0,1]$.

5. Find an example of a function which is not Lebesgue-summable, such that its improper Riemann integral exists and is finite.