Sheet 9

1. Let λ be the Lebesgue measure on $\mathbb{R}$ and f a non-negative, summable function on $(\mathbb{R}, \lambda)$. Show that the following equality of Lebesgue integrals holds:

$$\int_{\mathbb{R}} f d\lambda = \int_0^{+\infty} \lambda(\{f > s\}) ds.$$

Hint: In a first instance prove the equality when f is a simple function; in this case, make a picture of f and of the function $s \to \lambda(\{f > s\})$ and interpret the two sides according to the definition of Lebesgue-integral!

2. Take a Radon measure μ on $\mathbb{R}^n$ and let $\Omega \subset \mathbb{R}^n$ be a μ-measurable subset. Let $f, f_1, f_2, ...f_n...$ be non-negative summable functions such that $f_n \to f$ μ-a.e. and $\lim_n \int_\Omega f_n d\mu = \int_\Omega f d\mu$. Take a measurable set $E \subset \Omega$; show that

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu.$$

Hint: Use Fatou's lemma on E and on its complement E^c.

3. Let f_n be defined by ($n \in \mathbb{N}$):

$$f_n(x) = \frac{n\sqrt{x}}{1 + n^2x^2}, \quad x \in [0, 1].$$

Prove that:

(i) $f_n(x) \leq \frac{1}{\sqrt{x}}$ on $(0, 1]$ for all $n \geq 1$;

(ii) $\lim_{n\to\infty} \int_0^1 f_n(x) dx = 0.$

Bitte wenden!

4. Show the following result.

Let $\mathbb{R}^n$ be endowed with a Radon measure μ. Take a sequence $\{f_n\}$ of μ-measurable functions on $\mathbb{R}^n$ such that $f_n \to f$ in measure for a μ-measurable f. Suppose that for any $n \in \mathbb{N}$ there is a summable function g_n on $\mathbb{R}^n$ such that $|f_n(x)| \leq |g_n(x)|$ μ-a.e. and assume that

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |g_n - g| d\mu = 0$$

for a summable function g. Then the following holds:

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} |f_n - f| d\mu = 0.$$

Hint: Two possibilities for the proof:

- During the lectures the analogous result was proven to hold (as a consequence of Vitali's theorem) with the additional assumption of working on a subset $\Omega \subset \mathbb{R}^n$ of finite measure, $\mu(\Omega) < \infty$. Use this result.
- Use Exercise 1.

5. Alternative proofs of Young's inequality. Young's inequality states that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

for $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $a, b > 0$. Prove the inequality by taking the logarithm of both sides.

6. Let μ be a Radon measure on $\mathbb{R}^n$ and $f : \Omega \to \overline{\mathbb{R}}$ a μ-summable function. For $A \subset \Omega$ measurable, define:

$$\nu(A) = \int_A f d\mu.$$

Show that for $f \geq 0$ μ-a.e. ν is a Radon measure on $\mathbb{R}^n$.