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Numerische Mathematik I

ETH Zürich
D-MATH

Homework Problem Sheet 7

For some problems, parts of the solution are already given. Fill in the gaps and complete the proofs where you see a red band at the left margin.

Introduction. Splines, interpolation

Problem 7.1 Solving Systems of Equations for Periodic Splines

(7.1a) Show that the *Sherman-Morrison formula* holds true for an invertible Matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, i. e.

$$(\mathbf{A} + \mathbf{u}\mathbf{v}^\top)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^\top\mathbf{A}^{-1}}{1 + \mathbf{v}^\top\mathbf{A}^{-1}\mathbf{u}}.$$

Solution: It is easy to see that

$$\left(\mathbf{I} + \mathbf{w}\mathbf{v}^\top\right)^{-1} = \mathbf{I} - \frac{\mathbf{w}\mathbf{v}^\top}{1 + \mathbf{v}^\top\mathbf{w}}$$

is true since

(7.1b) Let $\mathbf{B}\mathbf{c} = \mathbf{d}$ be a system of equations in order to determine the coefficients $\mathbf{c} \in \mathbb{R}^n$ of a periodic spline in an analogous way as in [NMI, Eq. 3.53] in the script. Choose $\mathbf{u}$, $\mathbf{v}$ and a tridiagonal matrix $\mathbf{A}$ such that $\mathbf{B} = \mathbf{A} + \mathbf{u}\mathbf{v}^\top$. Describe an algorithm for solving $\mathbf{B}\mathbf{c} = \mathbf{d}$ with complexity $\mathcal{O}(n)$.

Solution: The matrix $\mathbf{B}$ has the structure

$$\mathbf{B} = \begin{pmatrix} \alpha_0 & \beta_1 & & & \beta_n \\ \beta_1 & \alpha_1 & \beta_2 & & \\ & \beta_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ \beta_n & & & \beta_{n-1} & \alpha_{n-1} \end{pmatrix}.$$

To split $\mathbf{B}$ into $\mathbf{A} + \mathbf{u}\mathbf{v}^\top$, where $\mathbf{A}$ is tridiagonal, we choose (for example) $\mathbf{u}$ and $\mathbf{v}$ as

Using this, we can first rewrite the equation $\mathbf{B}\mathbf{c} = \mathbf{d}$ to

We construct the following algorithm for solving $\mathbf{B}\mathbf{c} = \mathbf{d}$:

Summing this up, we can reach a total runtime of $\mathcal{O}(n)$.

Problem 7.2 Hermite Interpolation

We want to interpolate $f(x) = \cos x$ on $[0, \frac{\pi}{2}]$ in $x = 0$ by a polynomial degree two.

(7.2a) Compute the Lagrange interpolating polynomial $p_{2,\varepsilon}$ with supporting points $x_0 = 0$, $x_1 = \varepsilon$, $x_2 = \pi/2$. Then compute $p_{2,\varepsilon\to 0}(x) := \lim_{\varepsilon\to 0} p_{2,\varepsilon}(x)$.

Solution: We apply Newton's scheme to get

This implies that the interpolating polynomial is given by

For the limit $\varepsilon \to 0$ we use de l'Hôpital's rule for the coefficients to get

(7.2b) Compute the Hermite interpolating polynomial q_2 of degree two on the supporting points x_0, x_2 from subproblem (7.2a) only but using the first derivative of f, $f'(0)$, as interpolation data. Compare $p_{2,\varepsilon\to 0}$ to q_2.

Solution: Here, we apply Hermite's scheme, to get

Therefore, the Hermite interpolating polynomial is $q_2(x) = 1 - \frac{4}{\pi^2}x^2$. This shows by example that Hermite interpolation can be seen as a limiting process of Newton interpolation for two collapsing data points.

(7.2c) Now compute the Hermite interpolating polynomial $w_{2,\delta}$ of degree two for the data $(0, f(0), f'(0))$ and $(\delta, f(\delta))$. Then compute $w_{2,\delta\to 0}(x) := \lim_{\delta\to 0} w_{2,\delta}(x)$ again. Compare $w_{2,\delta}$ to the Taylor series of f around $x = 0$.

Solution: In this case, the Hermite scheme results in the table

Consequently, we get $w_{2,\delta}(x) = 1 + \frac{\cos(\delta)-1}{\delta^2}x^2$. For the limit $\delta \to 0$, we apply de l'Hôpital's rule on the coefficient to get

Problem 7.3 Trigonometric Interpolation

Fourier sums and polynomials are closely related as has become apparent in the proof of [NMI, Thm. 3.23]. In this problem we study interpolation by Fourier sums that are aptly called *trigonometric polynomials*. Hence, the title of this problem.

Let $f \in C^0(\mathbb{R})$ be a 2π-periodic function, that is $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$. Consider the interpolation nodes $x_j = 2\pi j/n$ for $j = 0, \ldots, n-1$ and $n = 2m + 1$ with $m \in \mathbb{N}$.

(7.3a) Show that there exists a unique vector $\mathbf{c} = (c_{-m}, \ldots, c_m) \in \mathbb{C}^n$ such that

$$q(x_j) = f(x_j) \quad \text{for } j = 0, \ldots, n-1 \quad \text{where} \quad q(t) = \sum_{k=-m}^{m} c_k \mathrm{e}^{\mathrm{i}kt}$$

HINT: Reduce to polynomial interpolation.

Solution: Let us evaluate the trigonometric polynomial $q(t)$ at the interpolation nodes, namely for $j = 0, \ldots, n-1$

$$q(x_j) =$$

(7.3b) What is the expression of the interpolant $q(t)$ from subproblem (7.3a) when $f(t) = \mathrm{e}^{\mathrm{i}\ell t}$ and $\ell \in \mathbb{Z}$?

Solution:

(7.3c) Let $f(t)$ be given as a so called "Fourier series"

$$f(t) = \sum_{\ell=-\infty}^{\infty} \widehat{f}_\ell \mathrm{e}^{\mathrm{i}\ell t} \quad , \quad \widehat{f}_\ell \in \mathbb{C} , \tag{7.3.1}$$

with $\left|\widehat{f}_\ell\right| \leq C\ell^{-2}$ for some $C > 0$ (Why this assumption?). Compute the corresponding trigonometric interpolant $q(t)$ as introduced in subproblem (7.3a).

HINT: Use subproblem (7.3b).

Solution: Based on subproblem (7.3b), we have the unique trigonometric interpolant of $t \mapsto \widehat{f}_\ell \mathrm{e}^{\mathrm{i}\ell t}$ is $t \mapsto \widehat{f}_\ell \mathrm{e}^{\mathrm{i}\ell' t}$ where $\ell = nv + \ell'$ for some $v \in \mathbb{Z}$ and $\ell' \in \{-m, \ldots, m\}$. Therefore, the coefficient $c_{\ell'}$ to $\mathrm{e}^{\mathrm{i}\ell' t}$ is given as

(7.3d) Finally, we tackle an interpolation error estimate for trigonometric interpolation based on the Fourier series representation (7.3.1) of the interpolant.

Find an estimate for the maximum norm of the interpolation error $\|f - q\|_{\infty,\mathbb{R}}$ when f and q are defined as in subproblem (7.3c) and $\left|\widehat{f}_\ell\right| \leq C\ell^{-r}$, $r \in \mathbb{N} \setminus \{1\}$, for some $C > 0$.

Solution:

$$\begin{aligned}
\|f - q\|_{\infty,\mathbb{R}} &= \sup_{t\in\mathbb{R}} \left| \sum_{\ell=-\infty}^{+\infty} \widehat{f}_\ell \mathrm{e}^{\mathrm{i}\ell t} - \sum_{k=-m}^{m} \sum_{v=-\infty}^{+\infty} \widehat{f}_{nv+k} \mathrm{e}^{\mathrm{i}kt} \right| \\
&= \sup_{t\in\mathbb{R}} \left| \sum_{|\ell|>m} \widehat{f}_\ell \mathrm{e}^{\mathrm{i}\ell t} + \sum_{k=-m}^{m} \left(\widehat{f}_k \mathrm{e}^{\mathrm{i}kt} - \sum_{v=-\infty}^{+\infty} \widehat{f}_{nv+k} \mathrm{e}^{\mathrm{i}kt} \right) \right| \\
&\leq
\end{aligned}$$

Problem 7.4 Condition of the Newton-Cotes Formulas

(7.4a) Set up a linear system of equations for the scaled weights α_j, $j = 0, \ldots, n$ of the closed Newton-Cotes formula $Q^{(n)}$ on the interval $[0, 1]$.

(7.4b) Write a MATLAB function `getNCWeights(n)` that solves the linear system from subproblem (7.4a) and returns the solution vector $\boldsymbol{\alpha} \in \mathbb{R}^{n+1}$ as a row vector.

Calculate the weights for $n = 1, \ldots, 10$. At what order do the first negative weights occur?

(7.4c) Calculate the absolute condition $\kappa_{\text{abs}}(Q^{(n)})$ of the Newton-Cotes formulas for $n = 1, 2, \ldots, 20$ on $[0, 1]$ and plot the condition in a diagram with logarithmic y-axis.

Listing 7.1: Testcalls for Problem 7.4

```
n = 1:10;
for i = 1:length(n)
    alp = getNCWeights(n(i));
    fprintf('\n n = %.2f:\t', n(i));
    fprintf('%.4f    ', alp);
end
fprintf('\n');
```

Listing 7.2: Output for Testcalls for Problem 7.4

```
>> test_call

 n = 1.00:     0.5000    0.5000
 n = 2.00:     0.1667    0.6667    0.1667
 n = 3.00:     0.1250    0.3750    0.3750    0.1250
 n = 4.00:     0.0778    0.3556    0.1333    0.3556    0.0778
 n = 5.00:     0.0660    0.2604    0.1736    0.1736    0.2604
    0.0660
 n = 6.00:     0.0488    0.2571    0.0321    0.3238    0.0321
    0.2571    0.0488
 n = 7.00:     0.0435    0.2070    0.0766    0.1730    0.1730
    0.0766    0.2070    0.0435
 n = 8.00:     0.0349    0.2077    -0.0327    0.3702    -0.1601
    0.3702    -0.0327    0.2077    0.0349
 n = 9.00:     0.0319    0.1757    0.0121    0.2159    0.0645
    0.0645    0.2159    0.0121    0.1757    0.0319
 n = 10.00:    0.0268    0.1775    -0.0810    0.4549    -0.4352
    0.7138    -0.4352    0.4549    -0.0810    0.1775    0.0268
```

Problem 7.5 Integral Representation of the Interpolation Error

In [NMI, Thm. 3.6] we found a representation for the error of polynomial interpolation of a function that relied on evaluating a derivative of the function at an unknown position (x_* in the statement of the theorem).

There is another family of error representation formulas for polynomial interpolation on an interval $[a, b]$ that are of the form ($f \in C^{n+1}([a, b])$)

$$(f - P_{\mathcal{N}} f)(t) = \int_a^b G_{\mathcal{N}}(t, \xi) f^{(n+1)}(\xi)\,\mathrm{d}\xi \;, \tag{7.5.1}$$

where $G_{\mathcal{N}} : [a, b]^2 \to \mathbb{R}$ is a suitable *kernel function*. In this problem we derive such a representation for the simple case of linear interpolation and use it for estimating the interpolation error.

(7.5a) Assume that $f \in C^2([0, 1])$ and $p \in \mathbb{P}_1$ with $p(0) = f(0),\ p(1) = f(1)$. Show that for $t \in [0, 1]$

$$(p - f)(t) = \int_0^1 G(t, \xi) f''(\xi)\,\mathrm{d}\xi, \tag{7.5.2}$$

where the kernel function is given by

$$G(t, \xi) = \begin{cases} (1 - t)\xi & 0 \le \xi < t \\ t(1 - \xi) & t \le \xi \le 1 \end{cases} \quad . \tag{7.5.3}$$

HINT: Use integration by parts.

(7.5b) Error representations according to (7.5.1) are very useful for obtaining error estimates in norms that involve integrals.

Let $-\infty < a < b < \infty$ and $f \in C^2([a, b])$. Assume that $p \in \mathbb{P}_1$ with $p(a) = f(a),\ p(b) = f(b)$. Use Equation 7.5.2 to show that

$$\|f - p\|_{L^2([a,b])} \le (b - a)^2 \|f''\|_{L^2([a,b])} \tag{7.5.4}$$

where the L^2-norm of a continuous function g on $[a, b]$ is defined by

$$\|g\|^2_{L^2([a,b])} := \int_a^b |g(\xi)|^2 d\xi.$$

HINT: First prove (7.5.4) on the interval $[0, 1]$, i.e. for $a = 0$ and $b = 1$. Then prove the general case by considering the function $\hat{f}(t) := f(a + t(b - a)) \in C^2([0, 1])$ for $f \in C^2([a, b])$. This technique is known as *scaling argument*.

(7.5c) Error representations like (7.5.1) also yield estimates in the maximum norm, though they may not be as sharp as those extracted from [NMI, Eq. (3.11)].

Show that $f(a) = f(b) = 0$ implies that

$$\|f\|_{L^\infty([a,b])} \le (b - a)^2 \|f''\|_{L^\infty([a,b])}.$$

HINT: Proceed similar as in (7.5b).

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MATLAB: Submit all files in the online system. Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

References

[NMI] Lecture Notes for the course "Numerische Mathematik I".

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