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Spring Term 2016

Numerische Mathematik I

ETH Zürich
D-MATH

Homework Problem Sheet 8

For some problems, parts of the solution are already given. Fill in the gaps and complete the proofs where you see a red band at the left margin.
Introduction. Quadrature, Tschebyscheff polynomials.

Problem 8.1 Error of Quadrature

Let $[a, b] \subset \mathbb{R}$ with $a < b$ be a bounded interval and let $f \in \mathcal{C}^2([a, b])$.

(8.1a) Show that there is a constant $C > 0$ independent of a, b and f, such that

$$\left| \int_a^b f(x)\, \mathrm{d}x - Q_{[a,b]}[f] \right| \leqslant C \cdot |b - a|^3 \cdot \|f''\|_{\mathcal{C}^0([a,b])},$$

where $Q_{[a,b]}[\cdot]$ denotes the rectangle method.

Solution: There are many different solutions, we present some of them here:

Using polynomial interpolation.

The rectangle method is symmetric, so polynomials of degree smaller than or equal to one are integrated exactly. Let p interpolate f at points $x_0 = \frac{a+b}{2}$ and $x_1 \in [a, b] \setminus \{x_0\}$. Then we have $\|\omega_2\|_{\mathcal{C}^0([a,b])} \leqslant |b - a|^2$ and we know that $Q_{[a,b]}[f] = \int_a^b p(x)\, \mathrm{d}x$.

By [NMI, Thm. 3.6], it follows that $|f(x) - p(x)| \leqslant \frac{1}{2}|b - a|^2 \|f''\|_{\mathcal{C}^0([a,b])}$. Estimating the absolute value of the integral then gives

Using Newton. Set $x_0 = \frac{1}{2}(a + b)$. Then by the theorem about quadrature error of the Midpoint rule in [NMI, Sec. 4.1.2], we have

Using Taylor. Set again $x_0 = \frac{1}{2}(a+b)$. We have $f(x) = f(x_0) + hf'(x_0) + \frac{1}{2}(x-x_0)^2 f''(\xi(x))$ hence

Using Gauss. The trapezoidal rule is exactly the Gaussian quadrature for $n = 0$. By [NMI, Thm. 4.18], we have

$$\left| \int_{-1}^{1} \widetilde{f}(t)\,\mathrm{d}t - 2\widetilde{f}(0) \right| \leqslant \frac{1}{3} \|\widetilde{f}''\|_{\mathcal{C}^0([-1,1])}.$$

Set $x_0 = \frac{1}{2}(a+b)$ again. With $\widetilde{f}(t) := f(x_0 + (b-x_0)t)$ we get

$$\frac{b-a}{2}\left(2f(x_0) - \int_a^b f(x)\,\mathrm{d}x\right) = 2\widetilde{f}(0) - \int_{-1}^{1} \widetilde{f}(t)\,\mathrm{d}t$$

and

$$\|\widetilde{f}''\|_{\mathcal{C}^0([-1,1])} = \frac{(b-a)^2}{2^2} \|f''\|_{\mathcal{C}^0([a,b])},$$

so the result follows.

(8.1b) For $f \in \mathcal{C}^0([0,1])$ and $h = (N-1)^{-1}$, let $T_h[f]$ denote the iterated trapezoidal rule on N equidistant sampling points in the interval $[0,1]$.

Show that for $f(x) := x^\alpha$, where $0 < \alpha < 1$, we have

$$\left| \int_0^1 f(x)\,\mathrm{d}x - T_h[f] \right| = \mathcal{O}(h^{\alpha+1}) \quad \text{for } h \to 0.$$

What changes if $f(x) = x^\alpha g(x)$, where $g \in \mathcal{C}^3([0,1])$?

Solution: Let $T_J^{(K)}[f]$ denote the trapezoidal rule with K equidistant sampling points on the interval J.

By [NMI, Thm. 4.4] with $h = 1/(N-1)$, we get that the trapezoidal rule satisfies

$$\left| \int_a^b f(x)\,\mathrm{d}x - T_{[a,b]}^{(N)}[f] \right| \leqslant \frac{h^2}{12}(b-a) \max_{x \in [a,b]} |f''(x)|$$

for $f \in \mathcal{C}^2([a,b])$ – but this is the crucial point: $f \notin \mathcal{C}^2([a,b])$, so this theorem cannot be applied. Even just applying it to an interval $[h,1]$ does not help because $f''(h) \propto h^{\alpha-2}$.

So let $[a_i, b_i] := h[i-1, i]$ for $i = 1, \dots, N-1$ be the subintervals of the trapezoidal rule on N equidistant sampling points. On each of the N subintervals $J = [a_i, b_i] = [b_i - h, b_i]$, $i = 2, \dots, N-1$, the trapezoidal rule satisfies

For $f(t) = t^\alpha$ with $0 < \alpha < 1$, we have $f'(t) = \alpha t^{\alpha-1}$ and $f''(t) = \alpha(\alpha-1)t^{\alpha-2}$. Hence f'' is a strictly increasing function on $(0,\infty)$ and so we have $f''(\xi) \leqslant C b_i^{\alpha-2}$ for all $\xi \in J$.

Summing this over all $N-1$ intervals gives

Problem 8.2 Order of Quadrature

(8.2a) Calculate the points $x_0, x_1 \in [-1, 1]$ and the weights $A, B \in \mathbb{R}$ for the quadrature rule

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx Af(x_0) + Bf(x_1)$$

such that the formula has the highest possible degree. What is that degree?

Solution: A quadrature formula on $n+1$ points is of maximum degree $2n+2$, in our case $(n=1)$ we get degree ≤ 4. The degree equals 4 if all monomials $1, x, x^2, x^3$ can be integrated exactly by our formula. This is the case if and only if

$$2 = A + B \tag{8.2.1}$$

$$0 = Ax_0 + Bx_1 \tag{8.2.2}$$

$$\frac{2}{3} = Ax_0^2 + Bx_1^2 \tag{8.2.3}$$

$$0 = Ax_0^3 + Bx_1^3 \tag{8.2.4}$$

From (8.2.2) and (8.2.4) we conclude $Bx_1(x_0^2 - x_1^2) = 0$. Hence one of the following cases must hold:

(8.2b) Calculate the *exact* result of $I = \int_1^5 \left|\frac{1}{2}x - \frac{3}{2}\right|^3 \mathrm{d}x$ by using the result we obtained in subproblem (8.2a) and by modifying I in such a way that the integrand becomes a polynomial.

Solution:

From our quadrature formula we get:

Problem 8.3 Iterated Quadrature Formulas

Let x_i, $i = 0, \ldots, n$, with $-\infty < x_0 < x_1 < \ldots < x_n < \infty$ be fixed real numbers in arithmetic progression and let f be a smooth function on the interval $[x_0, x_n]$.

(8.3a) Check that $T_{h/2} = \frac{1}{2}(T_h + M_h)$, where

$$T_h[f] = \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right) \quad \text{and} \quad M_h[f] = h\sum_{i=0}^{n-1} f(x_i + h/2)$$

are the iterated trapezoidal rule and the iterated rectangle method, respectively.

Solution: We have

$$T_h[f] = \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right)$$

$$M_h[f] = h\sum_{i=0}^{n-1} f(x_{i+1/2}) \quad \text{where} \quad x_{i+1/2} := \frac{x_i + x_{i+1}}{2}.$$

Hence

(8.3b) Explicitly calculate one step of Romberg's method with values $T_h[f]$ and $T_{h/2}[f]$. Rearrange the result to a quadrature formula of step width h.

Solution: For f sufficiently smooth, we get an asymptotic expansion of the form

$$T_h[f] = I + c_2 h^2 + \mathcal{O}(h^4) \quad \text{and} \quad T_{h/2}[f] = I + c_2 \frac{h^2}{4} + \mathcal{O}(h^4),$$

where I is the exact value of the integral. We now combine both terms linearly such that the terms with order h^4 cancel out. Set $T_{00} = T_h[f]$ and $T_{10} = T_{h/2}[f]$. Then we get

(8.3c) What is the asymptotic behaviour of the error of the new method?

Solution:

Problem 8.4 Chebychev Polynomials

In this problem you will meet a special set of polynomials that form a basis for the spaces $\mathbb{P}_{n-1}$ of polynomials of degree $< n$, $n \in \mathbb{N}$. With respect to an also special set of interpolation nodes this basis has rather desirable properties.

Throughout this problem, for $n \in \mathbb{N}_0$ we set

$$T_n(t) := \cos(n \arccos(t)) \ , \quad -1 \leq t \leq 1 \ ,$$

and for $n \in \mathbb{N}$

$$\mathcal{Z}_n := \left\{ x_k := \cos\big((k + \tfrac{1}{2})\tfrac{\pi}{n}\big), \ k = 0, \ldots, n-1 \right\} . \tag{8.4.1}$$

(8.4a) Read through section 3.8 of the lecture notes.

(8.4b) Show that $T_0(t) = 1$, $T_1(t) = t$, and

$$T_{n+1}(t) + T_{n-1}(t) = 2tT_n(t) \ , \quad n \geq 1 \ , \quad -1 \leq t \leq 1 \ . \tag{8.4.2}$$

(8.4c) Show that for $n \geq 1$ the derivatives of the functions T_n satisfy the recursion

$$2T_n(t) = \frac{1}{n+1}\,\frac{d}{dt}T_{n+1}(t) - \frac{1}{n-1}\,\frac{d}{dt}T_{n-1}(t)\ .$$

(8.4d) Show that T_n for $n \geq 1$, is a polynomial of degree n with leading coefficient 2^{n-1}.

(8.4e) Show that

$$\mathcal{Z}_n = \{t \in \mathbb{R} :\ T_n(t) = 0\}\ .$$

(8.4f) Write a MATLAB function

```
function y = evalT(n,t)
```

that computes $y_i := T_n(t_i)$, $n \in \mathbb{N}_0$, for arguments t_i, $i = 1, \dots, m$, passed in the row vector `t`. No special functions like $\cos$ and its inverse must be used.

The results are to be returned in the row vector `y`. What is the asymptotic computational effort of `evalT` for $n \to \infty$ and $m \to \infty$?

HINT: Use the recursion formula (8.4.2) for T_n.

(8.4g) We consider the interpolation problem with $V = \mathbb{P}_{n-1}$ and set of interpolation points $\mathcal{Z}_n$.

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be the interpolation matrix for the basis $\mathcal{B} = \{T_0, T_1, \dots, T_{n-1}\}$ of V (i.e. the $(j+1)$th column of $\mathbf{A}$ equals $(T_j(x_0), \dots, T_j(x_{n-1}))^\top$. Show that there is a regular diagonal matrix $\mathbf{D}$ such that $\mathbf{AD}$ is an orthogonal matrix.

(8.4h) Using the result of subproblem (8.4g), write an efficient (in terms of computational effort and memory!) MATLAB function

```
function c = chebcoeff(y)
```

that computes the coefficients c_j, $j = 0, \dots, n-1$, in the representation

$$p(t) = \sum_{j=0}^{n-1} c_j T_j(t)$$

of the polynomial interpolant $p \in \mathbb{P}_{n-1}$ through the points (x_i, y_i), $i = 0, \dots, n-1$, where the nodes x_i are defined in (8.4.1). The data y_i are passed in the row vector `y`, and the coefficients are returned in the row vector `c`.

(8.4i) Based on (8.4.2) write a MATLAB function

```
function y = chebsum(c,t)
```

that computes

$$y_i = \sum_{j=0}^{n-1} c_j T_j(t_i)$$

for $i = 1, \ldots, m$. The values t_i are the components of the row vector `t`, and the values y_i are made available in the row vector `y`.

What is the asymptotic computational effort of `chebsum` for $n \to \infty$ and $m \to \infty$?

Published on April 21, 2016.

To be submitted on May 3, 2016.

MATLAB: Submit all files in the online system. Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

References

[NMI] Lecture Notes for the course "Numerische Mathematik I".

Last modified on April 20, 2016