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Numerische Mathematik I

ETH Zürich
D-MATH

Homework Problem Sheet 10

For some problems, parts of the solution are already given. Fill in the gaps and complete the proofs where you see a red band at the left margin.

Introduction. Quadrature, fixed point iterations

Problem 10.1 Error of Simpson's Rule and Gaussian Quadrature

(10.1a) Show that the following holds for the error of Simpson's rule for $f \in \mathcal{C}^4([a,b])$:

$$\int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{1}{2}a + \frac{1}{2}b\right) + f(b)\right] = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$$

where $\xi \in [a,b]$ is a suitable intermediate value. For this purpose follow the idea of the proof of the second part of [NMI, Thm. 4.2] and give detailed reasons on why every step is correct.

Solution: Just as in the script we use the Newtonian description of the error of interpolation and obtain:

$$\mathsf{I}_{[a,b]}[f] - Q^{(2)}_{[a,b]}[f] =$$

Next we will replace the divided differences of the sampling point that occurs twice $(a+b)/2$ as in [NMI, Cor. 3.12]:

$$\mathsf{I}_{[a,b]}[f] - Q^{(2)}_{[a,b]}[f] =$$

where the point $\widehat{\xi}_x$ in general depends on x. Since the polynomial term in the integral does not switch sign we can use the first intermediate value theorem of integration:

(10.1b) Prove [NMI, Thm. 4.18]: For any $f \in \mathcal{C}^{2n+2}([-1,1])$ there exists a $\xi \in (-1,1)$ such that

$$\int_{-1}^{1} f(x)\,\mathrm{d}x - Q^{(n)}[f] = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\int_{-1}^{1}\prod_{j=0}^{n}(x-x_j)^2\,\mathrm{d}x$$

Solution:

Since $Q^{(n)}p = \mathsf{I}p$ for all $p \in \mathbb{P}_{2n+1}$, where I denotes the integral operator $\mathsf{I}[f] := \int_a^b f\,\mathrm{d}x$ we can chose the Hermite interpolation polynomial for the data

and we get:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x - Q^{(n)}[f] =$$

To estimate the error of interpolation we use [NMI, Thm. 3.6]:

Problem 10.2 Gauss-Hermite Quadrature

Quadrature formulas can also be used to approximate the values of improper integrals. This problem discusses an example.

Let $f : \mathbb{R} \to \mathbb{R}$ be integrable, we define $I_w[f] := \int_{-\infty}^{\infty} f(t)w(t)\,\mathrm{d}t$ for the *weight function* $w(t) := \mathrm{e}^{-t^2}$. For $n \in \mathbb{N}$, let f_n denote the monomial $f_n(t) := t^n$.

(10.2a) From the Analysis course we know $I_w[f_0] = I_w[1] = \sqrt{\pi}$. Using integration by parts, prove that

$$I_w[t^n] = \begin{cases} 0 & n = 1, 3, \ldots, \\ 2^{-n/2}\sqrt{\pi}(1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-1)) & n = 2, 4, \ldots. \end{cases}$$

Solution: For n odd, the integrand is an odd function, so the integral is zero. For n even, $n = 0, 2, 4, \ldots$, it follows from integration by parts that

$$\int_{-\infty}^{\infty} t^n \mathrm{e}^{-t^2}\,\mathrm{d}t =$$

Rewriting this for $n = 2, 4, \ldots$ and applying it inductively, results in

(10.2b) Determine a polynomial H_2 of degree 2 that satisfies $I_w[t^n H_2] = 0$ for $n = 0, 1$ and $I_w[H_2^2] = 1$, and $H_2(t) \overset{t\to\infty}{\longrightarrow} \infty$. This polynomial H_2 is called the *Hermite polynomial* of degree 2.

Solution: We set $H_2(t) = a_0 + a_1 t + a_2 t^2$ with unknown coefficients a_i, $i = 0, 1, 2$. Then we must have

(10.2c) Find the zeros t_0, t_1 of H_2 with $t_0 < t_1$.

Solution: The two zeros of H_2 are

(10.2d) Let $\{\ell_0, \ell_1\}$ be the Lagrange interpolation polynomials associated to the points $\{t_0, t_1\}$ (the roots of H_2 from subproblem (10.2c)). Approximate the integral $I_w[f]$ for $f(t) = \cos(t)$ using the Gauss-Hermite quadrature formula given by

$$Q_w^{(1)}[f] := \sum_{i=0}^{1} f(x_i)\alpha_i \approx I_w[f],$$

where the weights are defined by $\alpha_i := I_w[\ell_i]$, $i = 0, 1$. Compare the result to the exact value

$$\int_{-\infty}^{\infty} \cos(t) \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{\mathrm{e}^{1/4}}.$$

Solution: The quadrature weights are

and $\alpha_1 = \alpha_0$ for symmetry reasons. Hence

$$1.3804 \approx \frac{\sqrt{\pi}}{e^{1/4}} = \int_{-\infty}^{\infty} f(t) e^{-t^2}\, dt \approx \sum_{i=0}^{1} f(t_i)\alpha_i = \sqrt{\pi} \cos\left(\frac{1}{\sqrt{2}}\right) \approx 1.3475.$$

Problem 10.3 Fixed-point Iteration in 1D

Consider the fixed-point iteration $\Phi(x) = x$, $n = 0, 1, \ldots$ for $\Phi : \mathbb{R} \to \mathbb{R}$ given by

$$\Phi(x) = \arctan\left(\frac{x}{2}\right).$$

(10.3a) Show that there exists a unique solution x^* for the fixed-point interation of $\Phi(x)$ and that for $x^{(0)} \in \mathbb{R}$ the iterative method $x^{(k+1)} = \Phi(x^k),\ k = 0, 1, 2, \ldots$ converges to x^*.

HINT: Apply [NMI, Thm. 5.10].

Solution:

(10.3b) Determine an a priori error estimate for $k = 25$ iterations when $x^{(0)} = \pi$.

Solution: Noting that $x^{(1)} = \arctan(\frac{\pi}{2})$ and applying the appropriate error estimate from [NMI, Thm. 5.10] we have with $L := \sup_{x\in\mathbb{R}} |\Phi'(x)| = \frac{1}{2}$

(10.3c) What is the order/speed of convergence of the fixed-point interation? Write a MATLAB function `FPIteration.m` that simulates $n = 25$ steps of the fixed-point iteration $\Phi(x) = x$, and then plot the error convergence in a semi logarithmic plot (type `help semilogy`). You'll observe that your plot shows a straight line. Explain why.

Solution:

Listing 10.1: Solution for subproblem (10.3c)

```
function FPIteration
% Function to calculate the solution to arctan(x/2) = x using
% fixed-point iteration

x = pi;
nIterations = 25;

f = @(x) atan(x/2);

y = zeros(nIterations, 1);
y(1) = x;

for i = 1:25
    y(i+1) = f(y(i));
    fprintf('iteration: %d, x = %f\n', i, y(i+1));
end

err = y - y(nIterations+1);

figure(1);
grid on
semilogy(1:nIterations+1, err, '.-b');
title('Convergence 1-D fixed-point iteration');
```

```
xlabel('k');
ylabel('Rel. error');
end
```

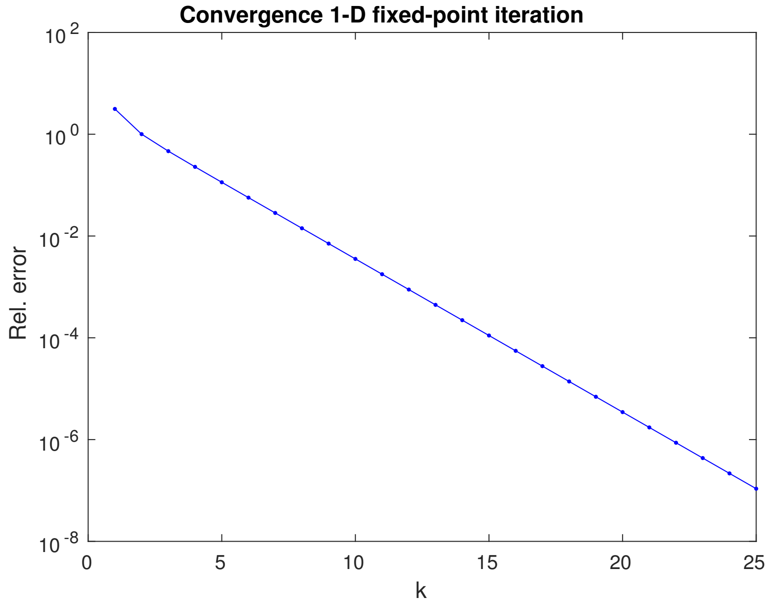

Figure 10.1: Fixed-point iteration for $\Phi(x) = \arctan(\frac{x}{2})$.

Problem 10.4 Gauss Quadrature Over General Interval

(10.4a) Implement a MATLAB function `GaussLegendre.m`, that takes as input the limits of integration `a` and `b`, the number points to use `n`$\in \mathbb{N}$ and the function handle `f`, and calculates the approximation $Q^{(n)}_{[a,b]}[f]$ to $\int_a^b f(x)\,\mathrm{d}x$ where $Q^{(n)}_{[a,b]}[f]$ denotes Gauss-Legendre quadrature. You may use the function `gaussQuad.m` given on the course website, to compute the quadrature points and weights for the interval $[-1, 1]$.

(10.4b) Plot the error convergence for $[a, b] = [-2, 2]$,

$$f_1(x) := \frac{|x|^{2.5}}{\sqrt{x^3 + 10}} \qquad \text{and} \qquad f_2(x) := \frac{x^2}{\sqrt{x^3 + 10}}$$

and $n = 1, \ldots, 80$. Explain.

Problem 10.5 Fixed-point Iteration in 2D

Consider the fixed-point iteration $\mathbf{z} = \Phi(\mathbf{z})$ for $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\Phi(\mathbf{z}) = \begin{pmatrix} \frac{1}{2}x^2 + \frac{1}{4}x + \frac{1}{2}y \\ \frac{2}{3}y \end{pmatrix}, \quad \text{where} \quad \mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}. \tag{10.5.1}$$

(10.5a) Calculate the Jacobian $\mathbf{J}$ of Φ and its inverse $\mathbf{J}^{-1}$.

(10.5b) Show that the fixed-point iteration of $\Phi(\mathbf{z})$ has a unique solution $\mathbf{z}^\star$ on

$$E = \left\{\mathbf{z} \in \mathbb{R}^2 \,\middle|\, x \in [-2/3, 1/5],\, y \in [-3/4, 1/4]\right\}$$

and that it converges in the ∞-norm for all initial values $\mathbf{z}^{(0)} \in E$.

HINT: Use Banach's fixed-point theorem.

(10.5c) Show that Φ is not a contraction on E in the 1-norm.

(10.5d) Let

$$\tilde{\Phi}(\mathbf{z}) := \frac{1}{7}\begin{pmatrix} 6\cos(x) - 2y \\ 2\sin(x) - \frac{2y}{1+y^2} \end{pmatrix}, \quad \text{where} \quad \mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Give a maximal set $\tilde{E} \subseteq \mathbb{R}^2$ s.t. the fixed point iteration $\tilde{\Phi}(\mathbf{z}^{(k)}) = \mathbf{z}^{(k+1)}$ converges for all initial values $\mathbf{z}^{(0)} \in \tilde{E}$.

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MATLAB: Submit all files in the online system. Include the files that generate the plots. Label all your plots. Include commands to run your functions. Comment on your results.

References

[NMI] Lecture Notes for the course "Numerische Mathematik I".

Last modified on May 7, 2016