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Numerical Analysis II

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D-MATH

Problem Sheet 5

Problem 5.1 Collocation vs. Quadrature rules

Observe the quadrature rule on $[0,1]$ of order p,

$$\sum_{i=0}^{s} b_i f(c_i) \approx \int_0^1 f(\xi)\,\mathrm{d}\xi,$$

with sample points c_i and weights b_i, $i=1,\dots,s$. So for all polynomials $f \in \mathcal{P}_p \setminus \mathcal{P}_{p-1}$

$$\sum_{i=0}^{s} b_i f(c_i) \neq \int_0^1 f(\xi) d\xi$$

holds.

The quadrature rule gives us a collocation method for the scalar differential equation $\dot{y} = f(t)$, $y(0) = y_0$ by

$$\begin{aligned} y_h(0) &= y_0 \\ y_h(t_{k+1}) &= y_h(t_k) + h \sum_{i=0}^{s} b_i f(hc_i). \end{aligned}$$

(5.1a) Show that the exact solution of the differential equation at the point h can be written as

$$y(h) = y_0 + h \int_0^1 f(\xi h)\,\mathrm{d}\xi.$$

(5.1b) Let f be sufficiently smooth. Show, that the explicit collocation method cannot be of order $p+1$.

HINT: Observe

$$\frac{\partial^n}{\partial h^n}\left(\frac{1}{h}\tau(0,y,h)\right)\Big|_{h=0}$$

where $\tau(0,y,h)$ is the consistency error after the first step

$$\tau(0,y,h) = (\Phi^{0,h} - \Psi^{0,h})y.$$

Problem 5.2 Linear Independence of Elementary Differentials

Show that the elementary differentials

$$f(\mathbf{y}_0), \quad Df(\mathbf{y}_0)f(\mathbf{y}_0), \quad Df(\mathbf{y}_0)Df(\mathbf{y}_0)f(\mathbf{y}_0), \quad D^2f(\mathbf{y}_0)(f(\mathbf{y}_0), f(\mathbf{y}_0))$$

(cf. example [NUMODE, Ex. 2.3.24]) are linearly independent as $\mathcal{C}^2(\mathrm{U}_\epsilon(\mathbf{y}_0), \mathbb{R}^2)$ mappings. In other words, show that if

$$\alpha_1 f(\mathbf{y}_0) + \alpha_2 Df(\mathbf{y}_0)f(\mathbf{y}_0) + \alpha_3 Df(\mathbf{y}_0)Df(\mathbf{y}_0)f(\mathbf{y}_0) + \alpha_4 D^2f(\mathbf{y}_0)(f(\mathbf{y}_0), f(\mathbf{y}_0)) = 0,$$

holds for all $f \in \mathcal{C}^2(\mathrm{U}_\epsilon(\mathbf{y}_0), \mathbb{R}^2)$, then it follows that $\alpha_i = 0$ for all $i = 1, \ldots, 4$.

Problem 5.3 The Implicit Midpoint Method for Non-Expansive Evolutions

The implicit midpoint method is a RK method and was previously introduced in [NODE, Ex. 2.1.5]. For the following autonomous initial value problem

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{y}_0. \tag{5.3.1}$$

the defining equations of the midpoint method are

$$\mathbf{y}_1 = \mathbf{y}_0 + h\mathbf{f}\big(\frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_0)\big). \tag{5.3.2}$$

By [NODE, Thm. 2.2.19] we know that (5.3.2) solvable with respect to $\mathbf{y}_1$ given that h is small enough and $\mathbf{f}$ is locally Lipschitz continuous.

We assume that $\mathbf{f} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a *dissipative vector field* ($\rightarrow$ [NUMODE, Def. 3.3.5], also [NODE, Rem. 2.2.16]) which satisfies a *global* Lipschitz condition.

The aim of this question is to show that the discrete evolution corresponding to the implicit midpoint method applied to (5.3.1) exists for every $h > 0$.

(5.3a) Convert (5.3.2) into the problem of finding the zeros of $\mathbf{G}(\mathbf{y}_1) = 0$ for an appropriate function $G : \mathbb{R}^d \mapsto \mathbb{R}^d$.

(5.3b) Show that for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^d$

$$\big(\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})\big)(\mathbf{x} - \mathbf{y}) \geq \|\mathbf{x} - \mathbf{y}\|^2,$$

in which $\mathbf{G}$ is the function from subproblem (5.3a).

(5.3c) Show that $\mathbf{G}$ is globally Lipschitz continuous on $\mathbb{R}^d$.

(5.3d) Prove the following Lemma, which guarantees the solvability of (5.3.2) with respect to $\mathbf{y}_1$.

Lemma: *Let* $\mathbf{G} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ *be a globally Lipschitz continuous function with the following property: there exists a* $\gamma > 0$, *such that*

$$\big(\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})\big)(\mathbf{x} - \mathbf{y}) \geq \gamma\|\mathbf{x} - \mathbf{y}\|^2, \tag{5.3.3}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. *Then* $\mathbf{G}(\mathbf{x}) = 0$ *has a unique solution.*
HINT: Let $\boldsymbol{\Phi}(\mathbf{x}) := \mathbf{x} - \alpha\mathbf{G}(\mathbf{x})$, for some $\alpha > 0$. First show that $\mathbf{x} \in \mathbb{R}^d$ is a unique solution of $\mathbf{G}(\mathbf{x}) = 0$ if and only if $\boldsymbol{\Phi}(\mathbf{x}) = \mathbf{x}$. Then show that for α small enough $\boldsymbol{\Phi}$ is a contraction on $\mathbb{R}^d$ and apply the Banach fixed-point theorem.

Problem 5.4 Implicit Trapezoidal Rule

The implicit trapezoidal rule for solving the autonomous differential equation $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ is given by

$$\mathbf{y}_1 = \Psi^{t_0,t_0+h} := \mathbf{y}_0 + \tfrac{1}{2}h[\mathbf{f}(\mathbf{y}_0) + \mathbf{f}(\mathbf{y}_1)]. \tag{5.4.1}$$

(5.4a) The method (5.4.1) can be interpreted as a Runge-Kutta-method. Produce the corresponding Butcher-tableau.

(5.4b) The method (5.4.1) can also be interpreted as a polynomial collocation method. What are the corresponding collocation points

Problem 5.5 Construction of One-Step Methods

This exercise studies the so called *Taylor series method* to construct one-step methods. For a given autonomous system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, we define a family of discrete evolutions $\Psi_n^{t,s}$, $n \in \mathbb{N}$ by

$$\Psi_n^{t,t+h}\mathbf{z} := \mathbf{z} + \sum_{i=1}^{n} \frac{h^i}{i!}\Delta_i \mathbf{z}, \qquad t, t+h \in J(\mathbf{z}),$$

where

$$\Delta_i \mathbf{z} := \frac{\mathrm{d}^{i-1}}{\mathrm{d}s^{i-1}}\mathbf{f}(\mathbf{y}(s))\Big|_{s=t} \qquad \text{with} \qquad \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{y}(s) = \mathbf{f}(\mathbf{y}(s)), \quad \mathbf{y}(t) = \mathbf{z}.$$

(5.5a) Show that the discrete evolutions $\Psi_n^{t,s}$ are consistent with the autonomous system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$.
HINT: Apply [NODE, Lemma. 2.1.10].

(5.5b) Express Δ_1, Δ_2 and Δ_3 symbolically in relation to the (sufficiently smooth) right hand side $\mathbf{f}$ and its derivatives.

(5.5c) Find the order of consistency $\boldsymbol{\tau}(t, \mathbf{y}, h)$ of these methods for autonomous systems.

(5.5d) We will now look at the scalar logistic differential equation

$$\dot{y} = 2y(1-y). \tag{5.5.1}$$

Write three MATLAB functions

```
function y = trv_1(h,T,y0)
function y = trv_2(h,T,y0)
function y = trv_3(h,T,y0)
```

in which, for a given step size `h`, end point `T` and initial value `y0`, you calculate the solution to (5.5.1) using Ψ_1, Ψ_2 and Ψ_3 respectively.

(5.5e) Write a MATLAB function

```
function p = trvconv
```

in which you calculate the convergence order of Ψ_1, Ψ_2 and Ψ_3 for (5.5.1).
HINT: Choose $y(0) = 2$ as initial value and calculate the error at the end point $T = 2$ for different step sizes $h = 2^{-4}, 2^{-5}, \ldots, 2^{-10}$. Then, apply a linear interpolation of the logarithm of the error vs. the logarithm of the stepsize with the MATLAB function `polyfit`.
HINT: The exact solution of (5.5.1) with initial value y_0 is $y(t) = y_0(y_0 + (1 - y_0)\mathrm{e}^{-2t})^{-1}$.

(5.5f) Formulate the discrete evolutions Ψ_n for the linear problem

$$\dot{\mathbf{y}} = \boldsymbol{A}\mathbf{y}, \qquad \mathbf{y} \in \mathbb{R}^d, \quad \boldsymbol{A} \in \mathbb{R}^{d\times d},$$

and general $n \in \mathbb{N}$. Implement Ψ_n, $n = 1, 2, 3$ for $\boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and test which of these methods is length-preserving.
HINT: First show, that $\Delta_i \mathbf{z} = \boldsymbol{A}^i \mathbf{z}$.

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References

[NODE] Lecture Notes for the course “Numerical Methods for Ordinary Differential Equations”.

[NUMODE] Lecture Slides for the course “Numerical Methods for Ordinary Differential Equations”, SVN revision # 52913.

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