Problem Sheet 10

Problem 10.1 Algebraic Stability

A Runge-Kutta 1-step method, represented by the Butcher-Tableau $\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^\top \end{array}$, is called *algebraically stable* provided its coefficients satisfy:

- $b_i \geq 0$, where $\mathbf{b} = (b_i)_{i=1}^s$,
- The matrix $\mathfrak{A}^\top \mathbf{D_b} + \mathbf{D_b}\mathfrak{A} - \mathbf{b}\mathbf{b}^\top$ is positive semi-definite.

where $\mathbf{D_b}$ denotes the diagonal matrix $\mathrm{diag}(b_1, \dots, b_s)$.

Algebraic stability is a sufficient condition for a discrete evolution of the Runge-Kutta method to inherit the non-expansivity (cf. [NODE, Def. 3.3.2] from the slides) of its corresponding continuous evolution. The advantage of this criterion is that it can be easily verified once the Butcher-Tableau is known.

(10.1a) Show that for a symmetric, positive semi-definite matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ and arbitrary vectors $\mathbf{u}_i \in \mathbb{R}^d,\ i = 1, \dots, n$, with $\mathbf{u}_i = (u_{ik})_{k=1}^d$ it holds that:

$$\sum_{i,j=1}^{n} m_{ij}\mathbf{u}_i^\top \mathbf{u}_j \geq 0, \quad \mathbf{M} = (m_{ij})_{i,j=1}^n$$

HINT: One way to show this is to use the fact that $\mathbf{M}$ can be orthogonally diagonalised by the principle axis transformation theorem.

(10.1b) Show that algebraic stability implies that the discrete evolution of a 1-step RK method, when applied to $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, is non-expansive if $\mathbf{f}$ is a dissipative vector field (see [NUMODE, Def. 3.3.5] in the slides, also notice [NODE, Lemma. 3.3.3]) with respect to the Euclidean vector norm (i.e. $\mathbf{M} = \mathbf{I}$).

HINT: Use the step form of the increment equations

$$\mathbf{y}_1 = \mathbf{y}_0 + h\sum_{i=1}^{s} b_i \mathbf{f}(\mathbf{g}_i) \qquad \text{with} \qquad \mathbf{g}_i = \mathbf{y}_0 + h\sum_{j=1}^{s} a_{ij}\mathbf{f}(\mathbf{g}_j), \tag{10.1.1}$$

and the fact that the Euclidean vector norm is induced by the Euclidean scalar product, to rewrite $\left\|\mathbf{y}_1 - \hat{\mathbf{y}}_1\right\|^2$ as the sum of $\left\|\mathbf{y}_0 - \hat{\mathbf{y}}_0\right\|^2$ and two further terms. Here $\hat{\mathbf{y}}_1$ denotes the value that comes from applying one step of the RK to the initial value $\hat{\mathbf{y}}_0$.

Use (10.1.1) to replace $\mathbf{y}_0 - \hat{\mathbf{y}}_0$ in the second term. Then use dissipativity of the underlying vector field and subproblem (10.1a) to finish the proof.

Problem 10.2 A-Stable Non-Expansive Method

We know from the lectures that we can deduce the A-stability of a 1-step RK method from the fact that it inherits the non-expansiveness of an evolution. The following problem serves to show that the converse does not hold.

Consider the following autonomous differential equation

$$\dot{y} = f(y) = \begin{cases} -y^3 & y < 0, \\ -y^2 & y \geq 0. \end{cases} \tag{10.2.1}$$

(10.2a) Show that the continuous evolution of (10.2.1) is non-expansive.

(10.2b) Show that the implicit trapezoidal method

$$\mathbf{y}_1 = \mathbf{y}_0 + \frac{h}{2}\mathbf{f}(\mathbf{y}_0) + \frac{h}{2}\mathbf{f}(\mathbf{y}_1)$$

is an A-stable Runge-Kutta method.

(10.2c) Show that the implicit trapezoidal method does not inherit the non-expansivity of the continuous evolution of (10.2.1).

HINT: Find $y_0 < 0$ such that $|y_1| > |y_0|$. Why does this imply that the method is not non-expansive?

Problem 10.3 Stability of Extrapolation Methods

Extrapolation is a valuable technique for the construction of single-step methods of higher order. Of particular interest is the extrapolation of the explicit Euler method whose stability properties we will now study.

(10.3a) Consider the autonomous initial value problem

$$y' = \lambda y \qquad y(0) = 1, \qquad \lambda \in \mathbb{C}. \tag{10.3.1}$$

Let h be the base stepsize. Show that after $n \in \mathbb{N}$ steps of the explicit Euler with step size $h_n = \frac{h}{n}$ the approximate solution is of the form

$$y(h) \approx y_n = \left(1 + \frac{h\lambda}{n}\right)^n.$$

(10.3b) Show that the stability function $S(z)$ of the extrapolated Euler method after k steps with extrapolation sequence $\{n_i\}_{i=1}^k$ is given by the recursive formula

$$\begin{aligned} S_{i,1}(z) &= \left(1 + \frac{z}{n_i}\right)^{n_i} && \text{for } i = 1, \dots, k, \\ S_{i,\ell}(z) &= S_{i,\ell-1}(z) + \frac{S_{i,\ell-1}(z) - S_{i-1,\ell-1}(z)}{\frac{n_i}{n_{i-\ell+1}} - 1} && \text{for } 2 \leq \ell \leq i, \\ S(z) &:= S_{k,k}(z). \end{aligned}$$

HINT: You can use the Aitken-Neville scheme [NUMODE, Eq. (2.4.5)], [NUMODE, Eq. (2.4.6)].

(10.3c) Write the MATLAB function

```
function s = StabilityEval(z, n)
```

in which given `z` (a complex number) and a vector of integers `n` we compute `s`, the value of the stability function S, of the extrapolated Euler as defined in (10.3b), at the point `z`.

(10.3d) Complete the MATLAB function `StabilityDomain.m`, which plots the stability domain of the extrapolated Euler method using 5 steps extrapolation steps (i.e. `n=1:5`). Comment on the stability properties of this method.

(10.3e) Prove that the extrapolated Euler method is not A-stable for any extrapolation sequence.

Problem 10.4 Ljapunov Functions

A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite* if $V(0) = 0$ and there exists an open ball $B = B(0, \epsilon)$ such that $V(x) > 0$ for all $x \in B$. The function V is called *positive semi-definite* if there exists a B such that $V(0) = 0$ and $V(x) \geqslant 0$ for all $x \in B$.

Positive definite quadratic forms, i.e. functions of the form $V(x) = x^T P x$ where $P \in \mathbb{R}^{n\times n}$, $P = P^T > 0$ are positive definite with respect to the above definition. Usually when studying stability problems, positive definite functions are called *Ljapunov functions*.

Let a Ljapunov function

$$\mathbf{V} : D \to \mathbb{R} \; .$$

be given on the state space $D \subset \mathbb{R}^n$.

(10.4a) Characterize that, for what kind of function $\mathbf{f} : D \to \mathbb{R}$ of an autonomous differential equation

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$$

on the state space D with the evolution $\boldsymbol{\Phi}$, the following relationship holds:

$$\mathbf{V}(\boldsymbol{\Phi}^t \mathbf{y}) \leq \mathbf{V}(\mathbf{y}) \qquad (10.4.1)$$

for all $\mathbf{y} \in D$ and all admissible $t \in \mathbb{R}$.

(10.4b) Show that, when the autonomous differential equation is discretized by Gauss methods, the relationship (10.4.1) of Ljapunov function still holds, i.e.

$$\mathbf{V}(\boldsymbol{\Psi}^h \mathbf{x}) \leq \mathbf{V}(\mathbf{x}) \; ,$$

for all $\mathbf{x} \in D$ and all admissible step sizes h.

HINT: Try to imitate the proof of [NODE, Thm. 4.1.5], and use subproblem (10.4a).

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References

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 52913.

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