Problem Sheet 10

Problem 10.1 Algebraic Stability

A Runge-Kutta 1-step method, represented by the Butcher-Tableau $\begin{array}{c|c} \mathbf{c} & \mathfrak{A} \\ \hline & \mathbf{b}^\top \end{array}$, is called *algebraically stable* provided its coefficients satisfy:

- $b_i \geq 0$, where $\mathbf{b} = (b_i)_{i=1}^s$,
- The matrix $\mathfrak{A}^\top \mathbf{D}_\mathbf{b} + \mathbf{D}_\mathbf{b}\mathfrak{A} - \mathbf{b}\mathbf{b}^\top$ is positive semi-definite.

where $\mathbf{D}_\mathbf{b}$ denotes the diagonal matrix $\operatorname{diag}(b_1, \dots, b_s)$.

Algebraic stability is a sufficient condition for a discrete evolution of the Runge-Kutta method to inherit the non-expansivity (cf. [NODE, Def. 3.3.2] from the slides) of its corresponding continuous evolution. The advantage of this criterion is that it can be easily verified once the Butcher-Tableau is known.

(10.1a) Show that for a symmetric, positive semi-definite matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ and arbitrary vectors $\mathbf{u}_i \in \mathbb{R}^d,\ i = 1, \dots, n$, with $\mathbf{u}_i = (u_{ik})_{k=1}^d$ it holds that:

$$\sum_{i,j=1}^n m_{ij}\mathbf{u}_i^\top \mathbf{u}_j \geq 0, \quad \mathbf{M} = (m_{ij})_{i,j=1}^n$$

HINT: One way to show this is to use the fact that $\mathbf{M}$ can be orthogonally diagonalised by the principle axis transformation theorem.

Solution: Let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be positive semi-definite. Then there exists a decomposition $\mathbf{M} = \mathbf{Q}^\top \mathbf{D}\mathbf{Q}$ with an orthogonal matrix $\mathbf{Q} = (q_{ij}) \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\mathbf{D} \in \mathbb{R}^{n\times n}$ with non-negative diagonal entries d_i. Therefore

$$\begin{aligned}\sum_{i,j=1}^n m_{ij}\mathbf{u}_i^\top \mathbf{u}_j &= \sum_{i,j=1}^n \left(\sum_{l=1}^n q_{li} d_l q_{lj}\right)\left(\sum_{k=1}^d u_{ik}u_{jk}\right) \\ &= \sum_{k=1}^d \sum_{l=1}^n d_l \Bigg(\underbrace{\sum_{i=1}^n (q_{li}u_{ik})}_{=:w_{l,k}} \underbrace{\sum_{j=1}^n (q_{lj}u_{jk})}_{=w_{l,k}}\Bigg) = \sum_{k=1}^d \sum_{l=1}^n d_l w_{l,k}^2 \geq 0.\end{aligned}$$

(10.1b) Show that algebraic stability implies that the discrete evolution of a 1-step RK method, when applied to $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, is non-expansive if $\mathbf{f}$ is a dissipative vector field (see [NUMODE, Def. 3.3.5] in the slides, also notice [NODE, Lemma. 3.3.3]) with respect to the Euclidean vector norm (i.e. $\mathbf{M} = \mathbf{I}$).

HINT: Use the step form of the increment equations

$$\mathbf{y}_1 = \mathbf{y}_0 + h\sum_{i=1}^{s} b_i \mathbf{f}(\mathbf{g}_i) \qquad \text{with} \qquad \mathbf{g}_i = \mathbf{y}_0 + h\sum_{j=1}^{s} a_{ij}\mathbf{f}(\mathbf{g}_j), \tag{10.1.1}$$

and the fact that the Euclidean vector norm is induced by the Euclidean scalar product, to rewrite $\|\mathbf{y}_1 - \hat{\mathbf{y}}_1\|^2$ as the sum of $\|\mathbf{y}_0 - \hat{\mathbf{y}}_0\|^2$ and two further terms. Here $\hat{\mathbf{y}}_1$ denotes the value that comes from applying one step of the RK to the initial value $\hat{\mathbf{y}}_0$.

Use (10.1.1) to replace $\mathbf{y}_0 - \hat{\mathbf{y}}_0$ in the second term. Then use dissipativity of the underlying vector field and subproblem (10.1a) to finish the proof.

Solution: We consider the increment equations

$$\mathbf{y}_1 = \mathbf{y}_0 + h\sum_{i=1}^{s} b_i \mathbf{f}(\mathbf{g}_i) \qquad \text{with} \qquad \mathbf{g}_i = \mathbf{y}_0 + h\sum_{j=1}^{s} a_{ij}\mathbf{f}(\mathbf{g}_j)$$

and

$$\hat{\mathbf{y}}_1 = \hat{\mathbf{y}}_0 + h\sum_{i=1}^{s} b_i \mathbf{f}(\hat{\mathbf{g}}_i) \qquad \text{with} \qquad \hat{\mathbf{g}}_i = \hat{\mathbf{y}}_0 + h\sum_{j=1}^{s} a_{ij}\mathbf{f}(\hat{\mathbf{g}}_j).$$

We want to show that, $\|\Delta\mathbf{y}_1\| := \|\mathbf{y}_1 - \hat{\mathbf{y}}_1\| \leq \|\mathbf{y}_0 - \hat{\mathbf{y}}_0\| =: \|\Delta\mathbf{y}_0\|$. Using the abbreviation $\Delta\mathbf{f}(\mathbf{g}_i) := \mathbf{f}(\mathbf{g}_i) - \mathbf{f}(\hat{\mathbf{g}}_i)$ we get

$$\begin{aligned}\|\Delta\mathbf{y}_1\|^2 &= \left\|\Delta\mathbf{y}_0 + h\sum_{i-1}^{s} b_i\Delta\mathbf{f}(\mathbf{g}_i)\right\|^2 \\ &= \|\Delta\mathbf{y}_0\|^2 + 2h\sum_{i=1}^{s} b_i\Delta\mathbf{f}(\mathbf{g}_i)\cdot\Delta\mathbf{y}_0 + h^2\left\|\sum_{i=1}^{s} b_i\Delta\mathbf{f}(\mathbf{g}_i)\right\|^2.\end{aligned}$$

In the middle term we substitute $\Delta\mathbf{y}_0$ by

$$\Delta\mathbf{y}_0 = \Delta\mathbf{g}_i - h\sum_{i=1}^{s} a_{ij}\Delta\mathbf{f}(\mathbf{g}_j)$$

with $\Delta\mathbf{g}_i = \mathbf{g}_i - \hat{\mathbf{g}}_i$ which yields

$$\|\Delta\mathbf{y}_1\|^2 = \|\Delta\mathbf{y}_0\|^2 + 2h\sum_{i=1}^{s} b_i\Delta\mathbf{f}(\mathbf{g}_i)\cdot\Delta\mathbf{g}_i - h^2\left[\sum_{i,j=1}^{s}(2b_i a_{ij} - b_i b_j)\Delta\mathbf{f}(\mathbf{g}_i)\cdot\Delta\mathbf{f}(\mathbf{g}_j)\right].$$

Because the vector field $\mathbf{f}$ is dissipative, we have $\Delta\mathbf{f}(\mathbf{g}_i)\cdot\Delta\mathbf{g}_i \leq 0$, and consequently

$$\begin{aligned}\|\Delta\mathbf{y}_1\|^2 &\leq \|\Delta\mathbf{y}_0\|^2 - h^2\left(\sum_{i,j=1}^{s}(\mathbf{D_b}\mathfrak{A} + \mathfrak{A}^\top\mathbf{D_b} - \mathbf{b}\mathbf{b}^\top)_{ij}\Delta\mathbf{f}(\mathbf{g}_i)\cdot\Delta\mathbf{f}(\mathbf{g}_j)\right) \\ &\overset{\textbf{(a)}}{\leq} \|\Delta\mathbf{y}_0\|^2,\end{aligned}$$

since $\mathbf{D_b}\mathfrak{A} + \mathfrak{A}^\top\mathbf{D_b} - \mathbf{b}\mathbf{b}^\top$ is positive semi-definite by assumption.

Problem 10.2 A-Stable Non-Expansive Method

We know from the lectures that we can deduce the A-stability of a 1-step RK method from the fact that it inherits the non-expansiveness of an evolution. The following problem serves to show that the converse does not hold.

Consider the following autonomous differential equation

$$\dot{y} = f(y) = \begin{cases} -y^3 & y < 0, \\ -y^2 & y \geq 0. \end{cases} \tag{10.2.1}$$

(10.2a) Show that the continuous evolution of (10.2.1) is non-expansive.

Solution: This immediately follows from the fact that f is a non-increasing function [NUMODE, Lemma 3.3.6]([NODE, Lemma. 3.3.3] is its special case).

(10.2b) Show that the implicit trapezoidal method

$$\mathbf{y}_1 = \mathbf{y}_0 + \frac{h}{2}\mathbf{f}(\mathbf{y}_0) + \frac{h}{2}\mathbf{f}(\mathbf{y}_1)$$

is an A-stable Runge-Kutta method.

Solution: The implicit trapezoidal method applied to the differential test equation

$$\dot{y} = \lambda y$$

results in

$$y_1 = y_0 + \frac{h}{2}\lambda y_0 + \frac{h}{2}\lambda y_1.$$

Therefore

$$\begin{aligned} y_1 &= \frac{2 + h\lambda}{2 - h\lambda} y_0 \\ &= S(h\lambda) y_0. \end{aligned}$$

On the imaginary axis $\{\mathrm{i}y \,|\, y \in \mathbb{R}\}$ it holds that

$$|S(\mathrm{i}y)| = 1.$$

In particular

$$S(\infty) = 1,$$

and $S(z)$ is holomorphic on $\mathbb{C}^-$. The A-stability of the implicit trapezoidal method is therefore a simple consequence of the maximum modulus principle for holomorphic functions (since S is clearly not a constant function).

(10.2c) Show that the implicit trapezoidal method does not inherit the non-expansivity of the continuous evolution of (10.2.1).

HINT: Find $y_0 < 0$ such that $|y_1| > |y_0|$. Why does this imply that the method is not non-expansive?

Solution: We want to find a $y_0 < 0$ such that $|y_1| > |y_0|$, since then

$$|\Psi(y_0) - \Psi(0)| = |y_1 - 0| > |y_0 - 0|.$$

would imply the claim of the problem.

First off, we transform the implicit trapezoidal method scheme

$$y_1 - \frac{h}{2}f(y_1) = y_0 + \frac{h}{2}f(y_0).$$

If $y_0 < 0$, then the equality becomes

$$y_1 - \frac{h}{2}f(y_1) = y_0 - \frac{h}{2}y_0^3.$$

If $y_0 < -\sqrt{\frac{2}{h}}$, then $y_1 > 0$, because

$$y_0 - \frac{h}{2}y_0^3 > 0 \qquad \text{for } y_0 < -\sqrt{\frac{2}{h}}$$

and

$$y_1 - \frac{h}{2}f(y_1) = y_1 + \frac{h}{2}y_1^3 < 0 \qquad \text{for } y_1 < 0.$$

Now we define

$$F(t) = t + \frac{h}{2}t^2 \qquad \text{for } t > 0,$$
$$G(t) = t - \frac{h}{2}t^3 \qquad \text{for } t < 0.$$

We observe that for $t < 0$

$$G(t) - F(-t) = t - \frac{h}{2}t^3 - \Big(-t + \frac{h}{2}(-t)^2 \Big)$$
$$= -t\Big(\frac{h}{2}t^2 + \frac{h}{2}t - 2\Big).$$

Hence $G(t) - F(-t) > 0$ for $t < -\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{16}{h}}$. Since F is non-decreasing,

$$F(y_1) = G(y_0) > F(-y_0)$$

implies that

$$y_1 > -y_0$$

for $y_0 < \min\big(-\frac{1}{2} - \frac{1}{2}\sqrt{1 + \frac{16}{h}}, -\sqrt{\frac{2}{h}}\big)$. Therefore using

$$|\Psi(y_0) - \Psi(0)| = |y_1 - 0| > |y_0 - 0|.$$

the implicit trapezoidal method is not non-expansive.

Problem 10.3 Stability of Extrapolation Methods

Extrapolation is a valuable technique for the construction of single-step methods of higher order. Of particular interest is the extrapolation of the explicit Euler method whose stability properties we will now study.

(10.3a) Consider the autonomous initial value problem

$$y' = \lambda y \qquad y(0) = 1, \qquad \lambda \in \mathbb{C}. \tag{10.3.1}$$

Let h be the base stepsize. Show that after $n \in \mathbb{N}$ steps of the explicit Euler with step size $h_n = \frac{h}{n}$ the approximate solution is of the form

$$y(h) \approx y_n = \left(1 + \frac{h\lambda}{n}\right)^n.$$

Solution: We shall use induction. A single step of the explicit Euler with step size $\frac{h}{n}$ yields

$$y_1 = y_0 + h_n f(y_0) = y_0 + \frac{h}{n}\lambda y_0 = 1 + \frac{h\lambda}{n}.$$

Let $m \leq n-1$. From the induction hypothesis

$$y_m = \left(1 + \frac{h\lambda}{n}\right)^m$$

it follows that

$$y_{m+1} = y_m + \frac{h}{n} f(y_m) = y_m + \frac{h}{n}\lambda y_m = \left(1 + \frac{h\lambda}{n}\right) y_m = \left(1 + \frac{h\lambda}{n}\right)^{m+1}.$$

Therefore, $y_n = \left(1 + \frac{h\lambda}{n}\right)^n$.

(10.3b) Show that the stability function $S(z)$ of the extrapolated Euler method after k steps with extrapolation sequence $\{n_i\}_{i=1}^k$ is given by the recursive formula

$$\begin{aligned} S_{i,1}(z) &= \left(1 + \frac{z}{n_i}\right)^{n_i} && \text{for } i = 1, \dots, k, \\ S_{i,\ell}(z) &= S_{i,\ell-1}(z) + \frac{S_{i,\ell-1}(z) - S_{i-1,\ell-1}(z)}{\frac{n_i}{n_{i-\ell+1}} - 1} && \text{for } 2 \leq \ell \leq i, \\ S(z) &:= S_{k,k}(z). \end{aligned}$$

HINT: You can use the Aitken-Neville scheme [NUMODE, Eq. (2.4.5)], [NUMODE, Eq. (2.4.6)].

Solution: Extrapolation via the Aitken-Neville scheme is admissible because the explicit Euler has consistency order 1. Let us now recall that by Aitken-Neville we have

$$\begin{aligned} T_{i,1} &:= y_{h_i}(x_0 + h) \\ T_{i,\ell} &:= T_{i,\ell-1} + \frac{T_{i,\ell-1} - T_{i-1,\ell-1}}{\frac{n_i}{n_{i-\ell+1}} - 1} && \text{for } 2 \leq \ell \leq i. \end{aligned}$$

Notice that n_i steps of a method that uses stepsize $\frac{h}{n_i}$ yields the approximation at $t_0 + h$. Hence, $S_{i,1}(z) = \left(1 + \frac{z}{n_i}\right)^{n_i}$ holds by subproblem (10.3a).

We will finish the proof by induction. Clearly, since $T_{i,l}$ is a linear combination of two methods, then its stability function is a linear combination of the stability functions of those two methods. In other words

$$\begin{aligned} T_{i,\ell} &= \left(1 + \frac{n_{i-\ell+1}}{n_i - n_{i-\ell+1}}\right) T_{i,\ell-1} + \frac{n_{i-\ell+1}}{n_i - n_{i-\ell+1}} T_{i-1,\ell-1} \\ &= \left(1 + \frac{n_{i-\ell+1}}{n_i - n_{i-\ell+1}}\right) S_{i,\ell-1}(z) y_0 + \frac{n_{i-\ell+1}}{n_i - n_{i-\ell+1}} S_{i-1,\ell-1}(z) y_0 \\ &= \left(S_{i,\ell-1}(z) + \frac{S_{i,\ell-1}(z) - S_{i-1,\ell-1}(z)}{\frac{n_i}{n_{i-\ell+1}} - 1} \right) y_0 \end{aligned}$$

Therefore, $S_{i,\ell}(z) = S_{i,\ell-1} + \frac{S_{i,\ell-1} - S_{i-1,\ell-1}}{\frac{n_i}{n_{i-\ell+1}} - 1}$, as was required.

(10.3c) Write the MATLAB function

```
function s = StabilityEval(z, n)
```

in which given `z` (a complex number) and a vector of integers `n` we compute `s`, the value of the stability function S, of the extrapolated Euler as defined in (10.3b), at the point `z`.

Solution:

Listing 10.1: Implementation of subproblem (10.3c)

```
function s = StabilityEval(z, n)

k=length(n);
S=(1+z./n).^n;

% In this implementation we will use the structure of the recursion,
% namely, the fact that recursion only uses the values in the current and
% previous row. This specific structure allows us to keep all of the
% important values in only a vector (instead of a matrix).
for l=2:k % Go over columns
    for i=2:k-l+2 % Go over rows
        % The recursion. Write over the preexisting value as we don't need
        % it any longer
        S(i-1)=S(i)+(S(i)-S(i-1))/(n(i+l-2)/n(i-1)-1);
    end
end

s=S(1);

% A straightforward implementation could look something like this
```

```
% Stemp=zeros(k,k);
% Stemp(:,1)=((1+z./n).^n)';
% for ll=2:k %for all columns
%     for ii=ll:k %for all lines
%
    Stemp(ii,ll)=Stemp(ii,ll-1)+(Stemp(ii,ll-1)-Stemp(ii-1,ll-1))...
%                /(n(ii)/n(ii-ll+1)-1);
%     end
% end
% Stemp(end,end);

end
```

(10.3d) Complete the MATLAB function `StabilityDomain.m`, which plots the stability domain of the extrapolated Euler method using 5 steps extrapolation steps (i.e. `n=1:5`). Comment on the stability properties of this method.

Solution: The method has a bounded stability domain, thus can't be A-stable.

Listing 10.2: Implementation of subproblem (10.3d)

```
function StabilityDomain

% Set the extrapolation sequence
n = 1:5;

% Build the grid using meshgrid
[X, Y] = meshgrid(-5:0.1:5);

% Generate complex mesh
Z = X + 1i*Y;

% Evaluate the stability function S on each point of the
    complex mesh
Sz = zeros(size(Z));
for ii = 1:length(Z)
    for jj = 1:length(Z)
        Sz(ii, jj) = StabilityEval(Z(ii, jj), n);
    end
end

% Plotting
figure;
% call contourf() with contour lines abs(S) = [0 1 2 5] here:
[~, h] = contourf(X, Y, abs(Sz), [0 1 2 5]);
set(h,'ShowText','on','TextStep',get(h,'LevelStep'));
colormap(autumn);
colorbar
```

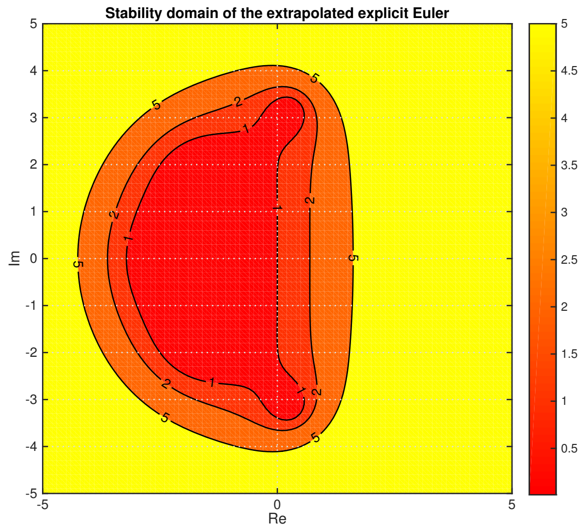

Figure 10.1: The stability domain as given by subproblem subproblem (10.3d)

```
title('Stability domain of the extrapolated explicit Euler');
xlabel('Re');
ylabel('Im');
axis square;
grid on;

print -depsc StabilityDomain
end
```

(10.3e) Prove that the extrapolated Euler method is not A-stable for any extrapolation sequence.

Solution: We have previously seen (Lemma 2.3.19 of the lecture notes) that extrapolation methods that are based on explicit methods are also explicit Runge-Kutta methods. Since explicit Runge-Kutta methods have a bounded stability domain, they cannot be A-stable and the conclusion follows.

Problem 10.4 Ljapunov Functions

A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called *positive definite* if $V(0) = 0$ and there exists an open ball $B = B(0, \epsilon)$ such that $V(x) > 0$ for all $x \in B$. The function V is called *positive semi-definite* if there exists a B such that $V(0) = 0$ and $V(x) \geqslant 0$ for all $x \in B$.

Positive definite quadratic forms, i.e. functions of the form $V(x) = x^T P x$ where $P \in \mathbb{R}^{n\times n}$, $P = P^T > 0$ are positive definite with respect to the above definition. Usually when studying stability problems, positive definite functions are called *Ljapunov functions*.

Let a Ljapunov function

$$\mathbf{V} : D \to \mathbb{R} \,.$$

be given on the state space $D \subset \mathbb{R}^n$.

(10.4a) Characterize that, for what kind of function $\mathbf{f} : D \to \mathbb{R}$ of an autonomous differential equation

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$$

on the state space D with the evolution $\boldsymbol{\Phi}$, the following relationship holds:

$$\mathbf{V}(\boldsymbol{\Phi}^t\mathbf{y}) \leq \mathbf{V}(\mathbf{y}) \tag{10.4.1}$$

for all $\mathbf{y} \in D$ and all admissible $t \in \mathbb{R}$.

Solution: Given is a Ljapunov function

$$\mathbf{V} : D \mapsto \mathbb{R}.$$

on the state space $D \subset \mathbb{R}^n$. If the right hand side $\mathbf{f} : D \mapsto \mathbb{R}$ of an autonomous differential equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

satisfies the condition $D\mathbf{V}(\mathbf{x}) \cdot \mathbf{f}(\mathbf{x}) \leq 0$, then the evolution $\boldsymbol{\Phi}$ satisfies the following relationship:

$$\mathbf{V}(\boldsymbol{\Phi}^t\mathbf{x}) \leq \mathbf{V}(\mathbf{x}) \tag{10.4.2}$$

for all $\mathbf{x} \in D$ and all admissible $t \in \mathbb{R}$.

(10.4b) Show that, when the autonomous differential equation is discretized by Gauss methods, the relationship (10.4.1) of Ljapunov function still holds, i.e.

$$\mathbf{V}(\boldsymbol{\Psi}^h\mathbf{x}) \leq \mathbf{V}(\mathbf{x}) \ ,$$

for all $\mathbf{x} \in D$ and all admissible step sizes h.

HINT: Try to imitate the proof of [NODE, Thm. 4.1.5], and use subproblem (10.4a).

Solution: Let us now show when the autonomous differential equation is discretized by Gauss collocation methods, the relationship (10.4.1) of Ljapunov function still holds, i.e.

$$\mathbf{V}(\boldsymbol{\Psi}^h\mathbf{x}) \leq \mathbf{V}(\mathbf{x})$$

for all $\mathbf{x} \in D$ an all admissible step sizes h.

For sufficiently small step size h, we have that $\mathbf{y}(h) := \boldsymbol{\Psi}^h(\mathbf{x})$ for fixed $x \in D$ is a polynomial $\mathbf{y} \in \mathcal{P}_s$ with $\mathbf{y}(0) = \mathbf{x}$. But then $q(\theta) := \mathbf{V}(\mathbf{y}(\theta h))$ is a polynomial $q \in \mathcal{P}_{2s}$ in θ. We then have

$$\begin{aligned}\mathbf{V}(\boldsymbol{\Psi}^h\mathbf{x}) = q(1) &= q(0) + \int_0^1 \dot{q}(\theta)\,\mathrm{d}\theta \\ &= \mathbf{V}(x) + \int_0^1 \dot{q}(\theta)\,\mathrm{d}\theta.\end{aligned}$$

We only need to show that the integral term is non-positive. The integral term can be determined exactly by Gaussian quadrature:

$$\int_0^1 \dot{q}(\theta)\,\mathrm{d}\theta = \sum_{j=1}^{s} b_j \dot{q}(c_j).$$

Because of the collocation condition $\dot{\mathbf{y}}(c_i h) = \mathbf{f}(\mathbf{y}(c_j h))$, we have:

$$\begin{aligned}\dot{q}(c_j) &= hD\mathbf{V}(\mathbf{y}(c_j h)) \cdot \dot{\mathbf{y}}(c_j h) \\ &= hD\mathbf{V}(\mathbf{y}(c_j h)) \cdot \mathbf{f}(\mathbf{y}(c_j h)) \leq 0, \quad 1 \leq i \leq s\end{aligned}$$

and, because the weights $b_j, 1 \leq j \leq s$ are all positive, we deduce that

$$\mathbf{V}(\boldsymbol{\Psi}^h \mathbf{x}) \leq \mathbf{V}(\mathbf{x}).$$

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References

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 52913.

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