Problem Sheet 12

Problem 12.1 Mathematical Pendulum

The mathematical Pendulum equation is given by

$$\begin{cases} \dot{q} &= p, \\ \dot{p} &= -\sin q. \end{cases} \tag{12.1.1}$$

(12.1a) Show that, (12.1.1) is a Hamilton system for the Hamilton function

$$H(p,q) = \frac{1}{2}p^2 - \cos q.$$

(12.1b) Show that, the energy $H(p,q)$ is a conserved quantity of (12.1.1).

(12.1c) Complete the MATLAB-template `pendelEuler.m` by solving (12.1.1) using the explicit Euler method. Plot the evolution of the energy as a function of time. What do you observe?

(12.1d) Complete the MATLAB-template `pendelMpr.m` by solving (12.1.1) using the implicit midpoint method and `nNewton` iterations of the Newton method. Execute the function for different values of `nNewton` and plot the evolution of the energy. What do you observe?

(12.1e) Complete the MATLAB-template `pendelMprKonv.m`, in which the convergence of the energy oscillations of the implicit midpoint method is investigated. The template solves (12.1.1) using the implicit midpoint for different time steps and saves the evolution of the energy in an array `energy`. You can read the maximal energy oscillation directly from this array, because the energy at the beginning is given. What do you observe?

Problem 12.2 Störmer-Verlet Method and Phase Volume

The volume in the phase space is an important invariant of the Hamilton's equations. Let $\boldsymbol{\Phi}^t$ be the continuous evolution of a Hamilton's equations and let $\mathbf{V}$ be some measurable subsets of the phase space. We identify these subsets as a set of initial values for the Hamilton's equations and define its time-based evolution as

$$\boldsymbol{\Phi}^t\mathbf{V} := \left\{ \boldsymbol{\Phi}^t\mathbf{y} \,\middle|\, \mathbf{y} \in \mathbf{V} \right\}.$$

The volume conservation in the phase space [NUMODE, Def. 4.2.1] is then given by

$$\int_{\boldsymbol{\Phi}^t\mathbf{V}} 1 \; \mathrm{d}\mathbf{x} = \int_{\mathbf{V}} 1 \, \mathrm{d}\mathbf{x} \quad \text{for all } t > 0. \tag{12.2.1}$$

(12.2a) Show that (12.2.1) holds if $|\det \mathbf{W}(t; t_0, \mathbf{y})| = 1$ for all $t > 0$ and for all $\mathbf{y} \in \mathbf{V}$, where $\mathbf{W}(t; t_0, \mathbf{y})$ is the Wronskian [NUMODE, Eq. (1.3.33)].

(12.2b) Now we want to numerically investigate, to what extent this important invariance is conserved by the Störmer-Verlet method. The Störmer-Verlet method is a symplectic method for solving differential equations of the form

$$\begin{cases} \dot{\mathbf{y}} & = \mathbf{v}, \\ \dot{\mathbf{v}} & = \mathbf{f}(t, \mathbf{y}), \end{cases} \tag{12.2.2}$$

whose continuous evolution is naturally volume conserving (see [NUMODE, Thm. 4.2.3]).

The Störmer-Verlet method is given by

$$\begin{aligned} \mathbf{v}_{k+1/2} &= \mathbf{v}_k + \frac{h}{2}\mathbf{f}(t_k, \mathbf{y}_k), \\ \mathbf{y}_{k+1} &= \mathbf{y}_k + h\mathbf{v}_{k+1/2}, \\ \mathbf{v}_{k+1} &= \mathbf{v}_{k+1/2} + \frac{h}{2}\mathbf{f}(t_{k+1}, \mathbf{y}_{k+1}). \end{aligned}$$

State the variational equation [NUMODE, Eq. (1.3.34)] for (12.2.2) and transform it so that it is of the same form as (12.2.2).

(12.2c) Complete the MATLAB template `stoermerverlet.m`, in which the initial value problem (12.2.2) and its variational equation for $\mathbf{f}(t, \mathbf{y}) : = -\sin(\mathbf{y})$ are solved simultaneously using the Störmer-Verlet method. Plot the evolution of the determinant of the Wronskian as a function of time. What do you observe?

(12.2d) Prove that the observation we can draw from subproblem (12.2c) is not unexpected. This of course implies that the Störmer-Verlet method conserves the phase volume even though the Wronskian is approximated by the Störmer-Verlet method.

HINT: The Störmer-Verlet method satisfies the claim of [NUMODE, Lem. 4.2.7] in the same sense as in subproblem (12.2c).

Problem 12.3 Hamiltonian Differential Equation

We consider the differential equation

$$\begin{pmatrix} \dot{p} \\ \dot{q} \end{pmatrix} = \frac{1}{p^2 + q^2} \begin{pmatrix} p \\ q \end{pmatrix}. \tag{12.3.1}$$

on $U := \mathbb{R}^2 \setminus \{0\}$.

(12.3a) Show that, the flow of the differential equation (12.3.1) is symplectic on every simply connected sub domain V of U.

HINT: For an appropriate definition of the logarithm (12.3.1) is a Hamiltonian differential equation on V with the Hamilton function $H(p, q) = -\operatorname{Im} \log(p + iq)$.

(12.3b) Why is (12.3.1) no Hamilton differential equation on the entire domain U?

(12.3c) Implement the symplectic Euler method [NUMODE, Eq. (4.4.27)] for (12.3.1) in a MATLAB function

```
[t,p,q]=SympEuler(p0,q0,tspan,h).
```

The initial value is given by `p0,q0`, range of integration `tspan` is of the form $[T_0, T_{\text{end}}]$, `h` is the step size, `t` is the time vector and `p,q` contain the approximated solutions at every time.

HINT: Use the MATLAB function `fsolve()` to solve non-linear systems of equations. The input and output parameters of this function are explained in `doc fsolve` or `help fsolve`

(12.3d) Implement the classical Runge-Kutta method of order 4 [NUMODE, Eq. (2.3.11)] for (12.3.1) in a MATLAB function

```
[t,p,q]=RK4(p0,q0,tspan,h).
```

The initial value is given by `p0,q0`, range of integration `tspan` is of the form $[T_0, T_{\text{end}}]$, `h` is the step size, `t` is the time vector and `p,q` contain the approximated solutions at every time.

(12.3e) Implement the MATLAB function `convergence.m` that uses (12.3.1) to determine the order of convergence of the symplectic Euler and the Runge-Kutta method in a numerical experiment. Use the start values $p(0) = 3$, $q(0) = 4$ and the stop time $T = 1$. The step size h should run through the values $2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$. What are the rates of convergence of both methods?

HINT: The analytic solution is $p(T) = 3\sqrt{\frac{2}{25}T + 1}, q(T) = 4\sqrt{\frac{2}{25}T + 1}$.

We consider the following Hamilton function $H(p, q) = -\operatorname{Im}\log(p + iq)$ to the initial value problem (12.3.1) with the initial values $p(0) = 3$, $q(0) = 4$. $\log$ is the logarithm function in MATLAB.

(12.3f) In a MATLAB file `energy.m` plot the trajectory of the Hamilton function $H(p, q) = -\operatorname{Im}\log(p + iq)$ for the solution of the symplectic Euler method and the RK4-method. Use the stop time $T = 100$ and the step size $h = \frac{1}{2}$. Also plot the trajectory of the numeric solution in the (p, q) plane.

(12.3g) In a MATLAB file `energyVar.m` investigate the oscillation of the Hamilton function $H(p, q) = -\operatorname{Im}\log(p + \mathrm{i}q)$ for the solutions of the Runge-Kutta method in dependence on the step size h. Use the stop time $T = 1$. The step size h should run through the values $2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}$. Plot the results.

(12.3h) Prove that, classical Runge-Kutta method of order 4 [NUMODE, Eq. (2.3.11)] conserves the Hamilton function corresponding to (12.3.1) accurately (in absence of rounding error).

HINT: $\operatorname{Im}\log(p + \mathrm{i}q) = \arg(p + \mathrm{i}q)$.

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References

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 52913.

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