Problem Sheet 13

Problem 13.1 A Volume Preserving Splitting Scheme

Consider the following ODE

$$\dot{\mathbf{y}} = f(\mathbf{y}) = \begin{pmatrix} -y_2 - \frac{y_1}{a^2+y_3^2} \\ y_1 - \frac{y_2}{a^2+y_3^2} \\ \frac{2\arctan\left(\frac{y_3}{a}\right)}{a} \end{pmatrix}, \quad \mathbf{y} \in \mathbb{R}^3, \quad a > 0. \tag{13.1.1}$$

(13.1a) Show that the flow of the ODE (13.1.1) is volume preserving.

On one hand, [NODE, Lemma. 4.2.5] dictates that there does not exist a Runge-Kutta scheme that is volume preserving for all problems in $\mathbb{R}^3$. On the other hand however, any SSM that preserves quadratic invariants is volume preserving in $\mathbb{R}^2$.

(13.1b) Split the vector field f, given as the right hand side of (13.1.1), as a sum of two-dimensional divergence-free vector fields, that is, as $f = f_1 + f_2 = \begin{pmatrix} f_1^1 \\ 0 \\ f_1^3 \end{pmatrix} + \begin{pmatrix} 0 \\ f_2^2 \\ f_2^3 \end{pmatrix}$.

HINT: Consider the construction in [NODE, Lemma. 4.2.6]

Now that we have the splitting $f(\mathbf{y}) = f_1(\mathbf{y}) + f_2(\mathbf{y})$ we can construct a volume preserving SSM. Take a Runge-Kutta SSM that preserves quadratic invariants, and denote by Ψ_i^h the flow of this SSM when applied to the ODE $\dot{\mathbf{y}} = f_i(\mathbf{y})$, $i = 1, 2$. The functions f_1 and f_2 are divergence free vector fields, and are (essentially) two-dimensional, while the flows Ψ_i^h are volume preserving. Hence, by applying a suitable splitting scheme (here we use Strang splitting) we can construct a volume preserving scheme

$$\Psi^h = \Psi_1^{h/2} \circ \Psi_2^h \circ \Psi_1^{h/2}$$

since the composition of volume preserving schemes is again volume preserving.

(13.1c) Finish the MATLAB code

```
function y = GaussStep(y0, f, Df, h)
```

which computes one step of the Gauss collocation scheme of order 4. The inputs are the initial value `y0`, the right hand side of the given ODE `f`, the derivative of the right-hand side `Df`, and the stepsize `h`. In order to compute the coefficients of the given Runge-Kutta method use the codes `collCoeffs.m` and `GaussNodes.m`, and in order to solve the underlying implicit system for the stages, use Newton's algorithm by completing the template `newton.m`.

(13.1d) Consider the inital value problem

$$\dot{\mathbf{y}} = \begin{pmatrix} -y_2 - \frac{y_1}{a^2+y_3^2} \\ y_1 - \frac{y_2}{a^2+y_3^2} \\ \frac{2\arctan\left(\frac{y_3}{a}\right)}{a} \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad a = 1.$$

Find its solution by using the previously described volume-preserving splitting scheme. Here Ψ_1^h and Ψ_2^h are both to be computed using steps of the Gauss collocation method of order 4. Complete the template `VolumePreservingSplitting.m` in which you should use `GaussStep.m` from (13.1c) in each step, and Strang splitting to compute the approximate solution.

Problem 13.2 The Symplectic Euler Method

We consider the Hamiltonian differential equation:

$$\dot{\mathbf{p}} = -\frac{\partial}{\partial \mathbf{q}} H(\mathbf{p}, \mathbf{q}), \qquad \dot{\mathbf{q}} = \frac{\partial}{\partial \mathbf{p}} H(\mathbf{p}, \mathbf{q}).$$

Show by direct calculations that, the so called symplectic Euler method from the lecture:

$$\mathbf{p}_1 = \mathbf{p}_0 - h\frac{\partial}{\partial \mathbf{q}} H(\mathbf{p}_1, \mathbf{q}_0), \qquad \mathbf{q}_1 = \mathbf{q}_0 + h\frac{\partial}{\partial \mathbf{p}} H(\mathbf{p}_1, \mathbf{q}_0),$$

is in fact symplectic.

Problem 13.3 Spring Pendulum

We consider the frictionless spring pendulum system with a massless spring, see [NUMODE, Ex. 4.4.35]. The Hamilton function:

$$H_K(\mathbf{p}, \mathbf{q}) = \frac{1}{2}(p_1^2 + p_2^2) + (\sqrt{q_1^2 + q_2^2} - 1)^2 + q_2$$

describes a spring pendulum with frictionless spring in cartesian coordinates i.e. $q_1 = x_1, q_2 = x_2$(Here x_2 axis points downwards). Notice here $\frac{1}{2}(p_1^2 + p_2^2)$ denotes the kinetic energy K of spring. In polar coordinates (r, φ), the kinetic energy K can be rewritten as

$$\begin{aligned} K &= \frac{1}{2}(p_1^2 + p_2^2) = \frac{1}{2}(\dot{x_1}^2 + \dot{x_2}^2) \\ &= \frac{1}{2}(\dot{r}^2 + r^2\dot{\varphi}^2). \end{aligned}$$

The generalized momentum conjugate to polar coordinates (r, φ) are

$$\begin{aligned} p_r &= \frac{\partial K}{\partial \dot{r}} = \dot{r}, \\ p_\varphi &= \frac{\partial K}{\partial \dot{\varphi}} = r^2\dot{\varphi}. \end{aligned}$$

Hence $K = \frac{1}{2}(p_r^2 + r^{-2}p_\varphi^2)$ in polar coordinates, and the Hamilton function is

$$H_P\left(\begin{pmatrix} p_r \\ p_\varphi \end{pmatrix}, \begin{pmatrix} r \\ \varphi \end{pmatrix}\right) = \frac{1}{2}(p_r^2 + r^{-2}p_\varphi^2) - r\sin(\varphi) + \frac{1}{2}(r-1)^2 \;.$$

(13.3a) Formulate an *explicit* symplectic Euler-method for

$$\dot{\mathbf{p}} = -\frac{\partial}{\partial \mathbf{q}} H_P(\mathbf{p}, \mathbf{q}), \quad \dot{\mathbf{q}} = \frac{\partial}{\partial \mathbf{p}} H_P(\mathbf{p}, \mathbf{q})$$

HINT: Write the symplectic Euler method in individual components $\mathbf{p}$ and $\mathbf{q}$.

(13.3b) Implement the symplectic Euler method from subproblem (13.3a) in MATLAB and solve the equation of motion for $\mathbf{p}_0 = 0$, $r_0 = 1$, $\varphi_0 = \pi/6$ for the time period $[0, 1000]$.

(13.3c) What observations do you make for large step sizes? Try to explain the observation.

(13.3d) Analyse the size of the oscillations of the total energy $H(\mathbf{p}, \mathbf{q})$ of the system during the numerical integration in dependence of the uniform time step size.

Published on 23 May 2016.

To be submitted by 31 May 2016.

References

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 52913.

Last modified on May 20, 2016