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Numerical Analysis II

ETH Zürich
D-MATH

Problem Sheet 2

Problem 2.1 Hamiltonian of the Lotka-Volterra Differential Equation

In [NODE, Eq. 1.4], we got to know the Lotka-Volterra differential equation,

$$\begin{aligned} \dot{u} &= u(v-2) \\ \dot{v} &= v(1-u) \end{aligned}$$

as an example for a differential equation with a non-trivial first integral $I(u, v) = \ln u - u + 2\ln v - v$. A further more in-depth structural quality of this differential equation is the subject of this exercise and will hopefully shine a new light on the invariant.

(2.1a) What differential equation are satisfied by the functions $p = \ln u$ and $q = \ln v$, where u and v are solutions of (2.1)?

(2.1b) Show that the differential equation found in subproblem (2.1a) is Hamiltonian (c.f. [NODE, Def. 1.2.3]) and give the corresponding Hamiltonian function $H(p, q)$.

HINT: Apply the transformation $p = \ln u$ and $q = \ln v$ to the invariant $I(u, v)$

Problem 2.2 Initial Value Problem With Cross Product

We will observe the initial value problem

$$\dot{\mathbf{y}} = f(\mathbf{y}) := \mathbf{a} \times \mathbf{y} + c\mathbf{y} \times (\mathbf{a} \times \mathbf{y}), \quad \mathbf{y}(0) = \tfrac{\sqrt{3}}{3}(1,1,1)^\top, \tag{2.2.1}$$

where $c > 0$ and $\mathbf{a} \in \mathbb{R}^3 \setminus \{0\}$.

Note: $\mathbf{x} \times \mathbf{y}$ denotes the cross product between the vectors $\mathbf{x}$ and $\mathbf{y}$. It satisfies $\mathbf{x} \times \mathbf{y} \perp \mathbf{x}$. In MATLAB, it is available as the function `cross(x,y)`.

(2.2a) Show that the solution of (2.2.1) exists for all points in time and that $\|\mathbf{y}(t)\| = 1, \forall t$ holds.

(2.2b) Is $\mathbf{a}$ an attractive fixed point of (2.2.1)? Explain your answer.

HINT: Use the result from subproblem (2.2a)

(2.2c) For $\mathbf{a} = (1, 0, 0)^\top$, (2.2.1) was solved with the standard MATLAB integrators `ode45` and `ode23s` up to the point $T = 10$ (default Tolerances). Explain the different dependence of the total number of steps from the parameter c observed in Figure 2.1.

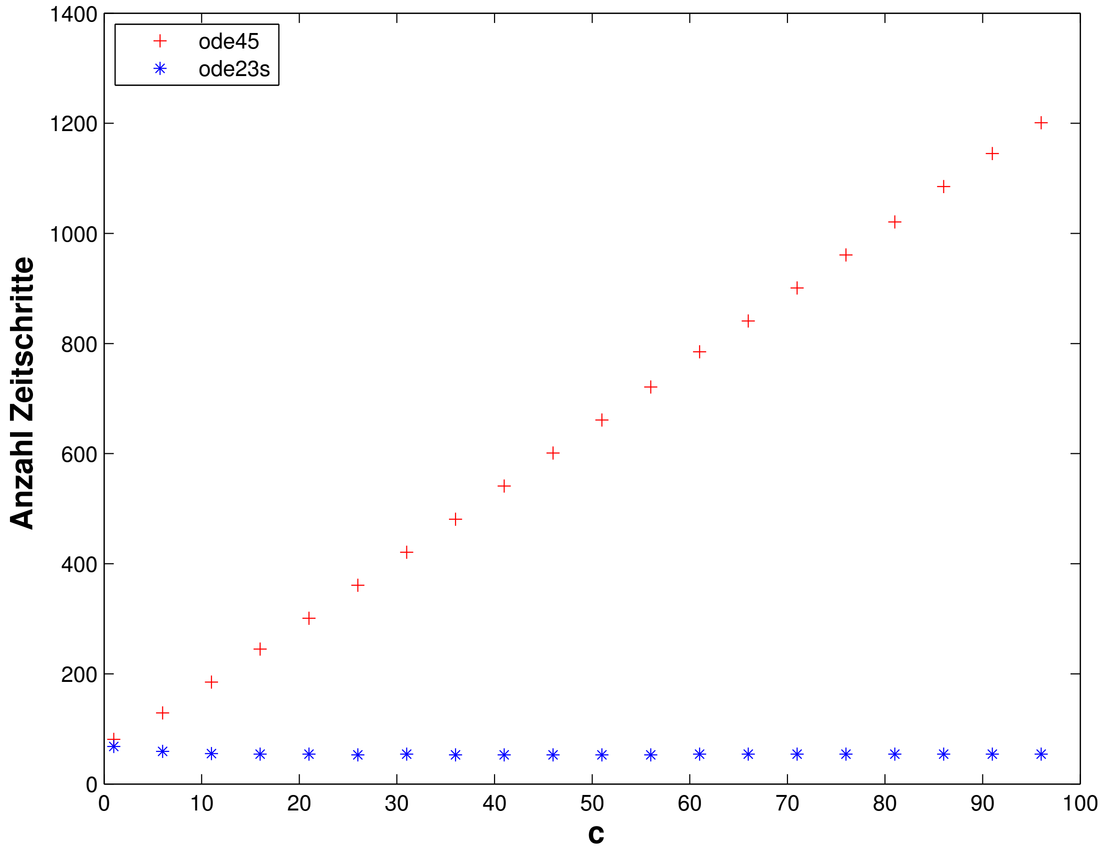

Figure 2.1: To subproblem (2.2c): number of steps used by standard MATLAB integrators in relation to the parameter c.

(2.2d) Formulate the non-linear equation given by the implicit mid-point rule for the initial value problem (2.2.1).

(2.2e) The linear-implicit mid-point rule for (2.2.1) can be received by a simple linearisation of the incremental equation of the implicit mid-point rule by the current solution value.

Give the defining equation of the linear-implicit mid-point rule for the general autonomous differential equation

$$\dot{\mathbf{y}} = f(\mathbf{y})$$

with smooth f.

(2.2f) Complete the implementation of the linear–implicit mid-point rule in the MATLAB–template `LinImpMidPointSolve.m`.

Solve (2.2.1) in the MATLAB–template `linimpmprsol.m` for the values $\mathbf{a} = (1, 0, 0)^\top$ and $c = 1$ as well as with `LinImpMidPointSolve.m` with step size $h = 0.1$ as well as with the MATLAB solver `ode45` up to the point $T = 5$. Plot the first component y_1 for both solutions.

Problem 2.3 Transforming the Thomas-Fermi Differential Equation

Let the *Thomas-Fermi* differential equation

$$\ddot{y}(t) = \frac{y^{\frac{3}{2}}(t)}{t^{\frac{1}{2}}}, \quad y(0) = y_0 \neq 0, \quad \dot{y}(0) = z_0 \tag{2.3.1}$$

be given. This form does not satisfy the local Lipschitz condition (cf.[NODE, Def. 1.3.4], [NODE, Ex. 1.3.5], [NODE, Thm. 1.3.6]) and is furthermore not suitable as an entry for numerical integration.

Transform (2.3.1) into a system of 1^{st} order that satisfies the Lipschitz condition.

HINT: Use the substitution

$$s = t^{\frac{1}{2}}, \quad y(t) = w(s), \quad u(s) = \frac{w'(s)}{s}.$$

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References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

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