Problem Sheet 3

Problem 3.1 Implementation and Study of the Explicit Euler Method

Observe the initial value problem

$$\dot{y} = f(t, y), \qquad y(0) = y_0,$$

where f will be defined later.

(3.1a) Complete the MATLAB template `EulerTemplate.m` for the explicit Euler method, by implementing the code for a Euler-step

$$y_h(t_{k+1}) = y_h(t_k) + hf(t_k, y_h(t_k)).$$

(3.1b) Now, observe the initial value problem

$$\dot{y} = -5y, \quad y(0) = 1.$$

Using the implementation from subproblem (3.1a), compute an approximation of the solution on the interval $[0, 5]$ with the step-sizes $h = 0.5$, $h = 0.25$ and $h = 0.1$. Plot the solutions, what do you observe? Explain the behavior.

(3.1c) Observe the initial value problem

$$\dot{y} = 50(y-1)^2(y-5), \quad y(0) = 1.1.$$

As in subproblem (3.1b), compute the approximation of the solution on the interval $[0, 0.4]$ with the step-sizes $h = 0.1$, $h = 0.05$ and $h = 0.01$. Plot the solutions, what do you observe? Explain the behavior once more.

Problem 3.2 Flow Map and Wronski Matrix

(3.2a) For the differential condition analysis of initial value problems with respect to perturbations of the initial data we have considered the differential of the solution and introduced it as the Wronskian in [NODE, Def. 1.3.20].

Prove the following property of the Wronskian for all admissible arguments t, s

$$\mathbf{W}(t; s, \boldsymbol{\Phi}^{t,s}\mathbf{y}_0)^{-1} = \mathbf{W}(s; t, \mathbf{y}_0).$$

HINT: [NODE, Def. 1.3.20] and the property $\boldsymbol{\Phi}^{s,t} \circ \boldsymbol{\Phi}^{t,s}\mathbf{y} = \mathbf{y}$ of flow maps.

(3.2b) Which of the three functions $\boldsymbol{\Phi}_i : \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2,\ 1 \le i \le 3$ where

(i) $\boldsymbol{\Phi}_1(t, \mathbf{y}) := \boldsymbol{\Phi}_1^t \mathbf{y} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \mathbf{y},$

(ii) $\boldsymbol{\Phi}_2(t, \mathbf{y}) := \boldsymbol{\Phi}_2^t \mathbf{y} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mathbf{y},$

(iii) $\boldsymbol{\Phi}_3(t, \mathbf{y}) := \boldsymbol{\Phi}_3^t \mathbf{y} = \begin{pmatrix} \exp(\lambda t) & t \\ 0 & \exp(\lambda t) \end{pmatrix} \mathbf{y}, \quad \lambda \in \mathbb{R} - \{0\}$

satisfy the group property $\boldsymbol{\Phi}_i^{t+s} = \boldsymbol{\Phi}_i^s \circ \boldsymbol{\Phi}_i^t$ (see [NODE, Lemma. 1.3.11])? Which functions can be interpreted as flow maps [NODE, Def. 1.3.10] of an autonomous differential equation $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, and which can't? Determine for the former ones the corresponding differential equations.

Problem 3.3 Discrete Gronwall Lemma

Prove the discrete Gronwall Lemma for constant h:

If the sequence $(\xi_k)_{k \in \mathbb{N}_0}$, $\xi_k \ge 0$ satisfies the inequality

$$\xi_{k+1} \le Ch^{p+1} + (1 + Lh)\xi_k, \quad k \in \mathbb{N}_0, \quad C, h \ge 0,\ L > 0,$$

then

$$\xi_k \le Ch^p \frac{1}{L}\left(e^{kLh} - 1\right) + e^{kLh} \cdot \xi_0, \quad k \in \mathbb{N}_0.$$

HINT: Show, by induction, that

$$\xi_k \le \frac{Ch^p}{L}\left[(1 + Lh)^k - 1\right] + (1 + Lh)^k \xi_0$$

and use the convexity of the exponential function.

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References

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

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