P. Grohs
W. Wu

Spring Term 2016

Numerical Analysis II

ETH Zürich
D-MATH

Problem Sheet 3

Problem 3.1 Implementation and Study of the Explicit Euler Method

Observe the initial value problem

$$\dot{y} = f(t, y), \qquad y(0) = y_0,$$

where f will be defined later.

(3.1a) Complete the MATLAB template `EulerTemplate.m` for the explicit Euler method, by implementing the code for a Euler-step

$$y_h(t_{k+1}) = y_h(t_k) + hf(t_k, y_h(t_k)).$$

Solution: See Listing 3.1

Listing 3.1: Completed Template from subproblem (3.1a)

```
function [t,y] = Euler(f,T,y0,h)
% Solves the initial value problem y'=f(t,y), y(0)=y0 up to time T
% using the explicit Euler metho and constant step size h.
%
% example:
% 1. With anonymous function for the right hand side of the diff. eq.:
%
% [t,y] = Euler(@(t,y) -5*y,5,1,0.1);
%
% 2. With a function for the right hand side defined in the MATLAB shell:
%
% g = @(t,y) -5*y;
% [t,y] = Euler(g,5,1,0.1);
%
% 3. With a function defined in its own MATLAB file "f.m":
%
% [t,y] = Euler(@f,5,1,0.1);
%
% Plot of the solution:
```

```
% plot(t,y,'x-')

  t(1) = 0;
  y(1) = y0;

  N = round((T-0)/h);

  for n=1:N
    y(n+1) = eulerstep(f,t(n),y(n),h);
    t(n+1) = t(n) + h;
  end

function yn = eulerstep(f,t,y,h)
  yn = y + h*f(t,y);
```

(3.1b) Now, observe the initial value problem

$$\dot{y} = -5y, \quad y(0) = 1.$$

Using the implementation from subproblem (3.1a), compute an approximation of the solution on the interval $[0, 5]$ with the step-sizes $h = 0.5$, $h = 0.25$ and $h = 0.1$. Plot the solutions, what do you observe? Explain the behavior.

Solution: With step-size $h = 0.5$, the numerical solution oscillates, with the oscillations even growing in time, see Figure 3.1. Even though the numerical solution with $h = 0.25$ is a poor approximation of the exact solution $y(t) = \mathrm{e}^{-5t}$, its qualitative behavior is appropriate. With step-size $h = 0.1$, the numerical solution is acceptable. The reason for the poor result for $h = 0.5$ is that the method with this step-size is *unstable* for the given equation. Stability will be discussed later in the lecture: See [NODE, Ch. 3] in the lecture notes.

(3.1c) Observe the initial value problem

$$\dot{y} = 50(y-1)^2(y-5), \quad y(0) = 1.1.$$

As in subproblem (3.1b), compute the approximation of the solution on the interval $[0, 0.4]$ with the step-sizes $h = 0.1$, $h = 0.05$ and $h = 0.01$. Plot the solutions, what do you observe? Explain the behavior once more.

Solution: Note, that $y \equiv 1$ and $y \equiv 5$ are constant solutions of the equation

$$\dot{y} = 50(y-1)^2(y-5).$$

From the uniqueness theorem, it follows that no solution ever reaches the lines $y(t) = 1$ or $y(t) = 5$. If y_0 lies in the interval $1 < y_0 < 5$, the solution will decrease in time, as $f(t, y) < 0$, but without reaching 1. As a consequence, the solution curves for $1 < y_0 < 5$ will always get closer to one another. If $y_0 < 1$, the solution will also decrease ($f(t, y) < 0$), but it is not bounded from below and will keep decreasing at a growing rate. From this, it follows that the two solutions for different initial values $y_0, \tilde{y}_0 < 1$ will grow apart, see Figure 3.2.

If we take $h = 0.1$, we come from a stable to an unstable area, so that the numerical solution approaches $-\infty$. For $h = 0.05$, the qualitative behavior of the solution is correct, although the resolution is very bad. The step-size $h = 0.01$ gives an acceptable solution. See Figure 3.3.

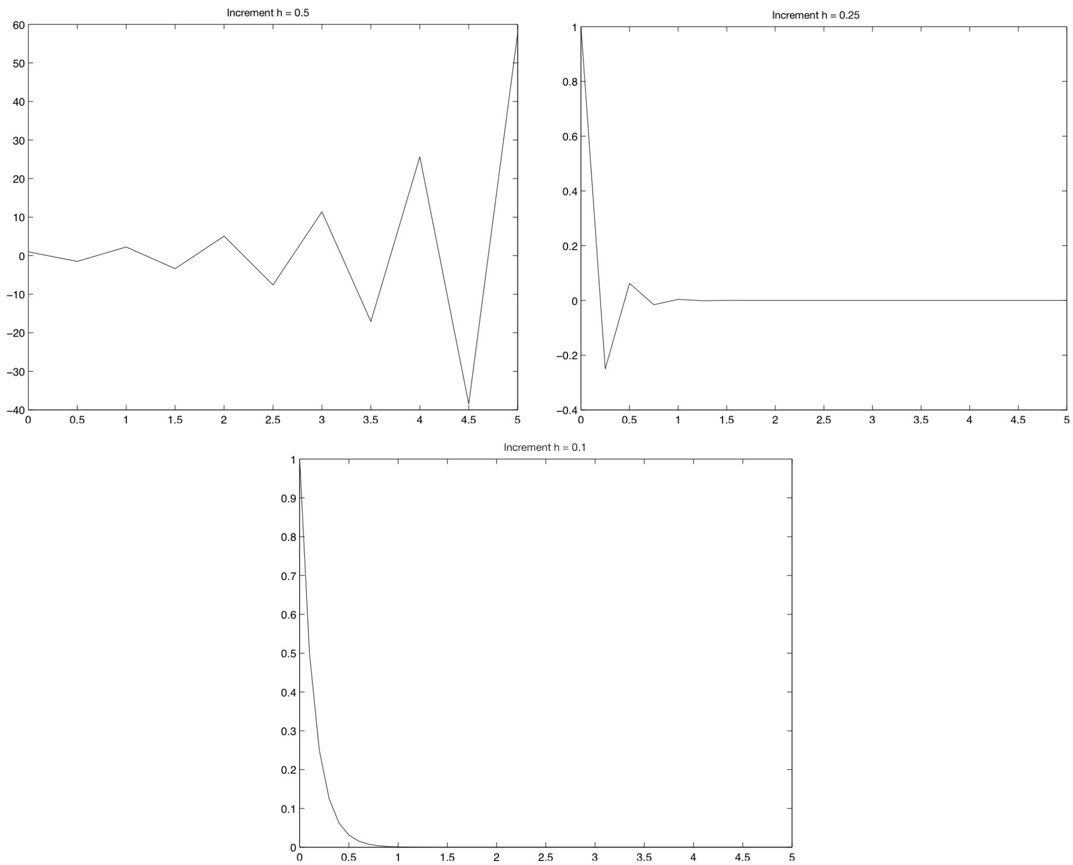

Figure 3.1: subproblem (3.1b), solutions for the equation $\dot{y} = -5y$.

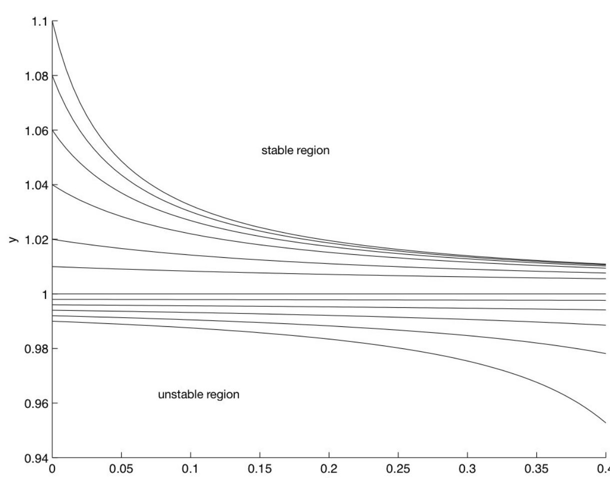

Figure 3.2: subproblem (3.1c), behavior of the solutions of $\dot{y} = 50(y-1)^2(y-5)$.

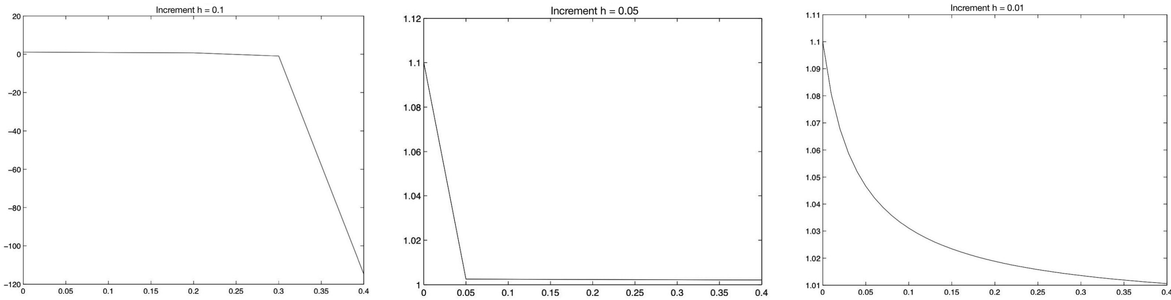

Figure 3.3: subproblem (3.1c), numerical solution for $\dot{y} = 50(y-1)^2(y-5)$.

Problem 3.2 Flow Map and Wronski Matrix

(3.2a) For the differential condition analysis of initial value problems with respect to perturbations of the initial data we have considered the differential of the solution and introduced it as the Wronskian in [NODE, Def. 1.3.20].

Prove the following property of the Wronskian for all admissible arguments t, s

$$\mathbf{W}(t; s, \boldsymbol{\Phi}^{t,s}\mathbf{y}_0)^{-1} = \mathbf{W}(s; t, \mathbf{y}_0).$$

HINT: [NODE, Def. 1.3.20] and the property $\boldsymbol{\Phi}^{s,t} \circ \boldsymbol{\Phi}^{t,s}\mathbf{y} = \mathbf{y}$ of flow maps.

Solution: For the continuous flow we have

$$\mathsf{Id} = \boldsymbol{\Phi}^{s,t} \circ \boldsymbol{\Phi}^{t,s}.$$

We differentiate

$$\mathbf{y} = \boldsymbol{\Phi}^{s,t} \circ \boldsymbol{\Phi}^{t,s}\mathbf{y}$$

with respect to $\mathbf{y}$ and obtain

$$\begin{aligned} I &= \left(\frac{\partial \boldsymbol{\Phi}^{s,t}}{\partial \boldsymbol{\Phi}^{t,s}} \frac{\partial \boldsymbol{\Phi}^{t,s}}{\partial \mathbf{y}} \mathbf{y} \right) \Big|_{\mathbf{y}=\mathbf{y}_0} \\ &= \frac{\partial \boldsymbol{\Phi}^{s,t}}{\partial \mathbf{y}} \Big|_{\mathbf{y}=\boldsymbol{\Phi}^{t,s}\mathbf{y}_0} \frac{\partial \boldsymbol{\Phi}^{t,s}}{\partial \mathbf{y}} \mathbf{y} \Big|_{\mathbf{y}=\mathbf{y}_0} \\ &= \mathbf{W}(t; s, \boldsymbol{\Phi}^{t,s}\mathbf{y}_0)\mathbf{W}(s; t, \mathbf{y}_0), \end{aligned}$$

hence

$$\mathbf{W}(t; s, \boldsymbol{\Phi}^{t,s}\mathbf{y}_0)^{-1} = \mathbf{W}(s; t, \mathbf{y}_0).$$

Once again in terms of components:

$$\begin{aligned} \delta_{i,j} &= \frac{\partial}{\partial y_i} \boldsymbol{\Phi}_j^{s,t}(\boldsymbol{\Phi}^{t,s}(y)) \\ &= \sum_k \partial_k \boldsymbol{\Phi}_j^{s,t}(\boldsymbol{\Phi}^{t,s}(y)) \frac{\partial}{\partial y_i} \boldsymbol{\Phi}_k^{t,s}(y) \\ &= \sum_k \mathbf{W}_{jk}(t; s, \boldsymbol{\Phi}^{t,s}(y))\mathbf{W}_{ki}(t; s, y). \end{aligned}$$

(3.2b) Which of the three functions $\boldsymbol{\Phi}_i : \mathbb{R} \times \mathbb{R}^2 \mapsto \mathbb{R}^2,\ 1 \leq i \leq 3$ where

(i) $\boldsymbol{\Phi}_1(t, \mathbf{y}) := \boldsymbol{\Phi}_1^t \mathbf{y} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \mathbf{y},$

(ii) $\boldsymbol{\Phi}_2(t, \mathbf{y}) := \boldsymbol{\Phi}_2^t \mathbf{y} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \mathbf{y},$

(iii) $\boldsymbol{\Phi}_3(t, \mathbf{y}) := \boldsymbol{\Phi}_3^t \mathbf{y} = \begin{pmatrix} \exp(\lambda t) & t \\ 0 & \exp(\lambda t) \end{pmatrix} \mathbf{y}, \quad \lambda \in \mathbb{R} - \{0\}$

satisfy the group property $\boldsymbol{\Phi}_i^{t+s} = \boldsymbol{\Phi}_i^s \circ \boldsymbol{\Phi}_i^t$ (see [NODE, Lemma. 1.3.11])? Which functions can be interpreted as flow maps [NODE, Def. 1.3.10] of an autonomous differential equation $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, and which can't? Determine for the former ones the corresponding differential equations.

Solution: The function $\boldsymbol{\Phi}_3$ does not satisfy the group property, since when $\lambda \neq 0$,

$$\begin{aligned} \boldsymbol{\Phi}_3^s \circ \boldsymbol{\Phi}_3^t &= \begin{pmatrix} \exp(\lambda s) & s \\ 0 & \exp(\lambda s) \end{pmatrix} \begin{pmatrix} \exp(\lambda t) & t \\ 0 & \exp(\lambda t) \end{pmatrix} \\ &= \begin{pmatrix} \exp(\lambda t + \lambda s) & s\exp(\lambda t) + t\exp(\lambda s) \\ 0 & \exp(\lambda t + \lambda s) \end{pmatrix} \\ &\neq \begin{pmatrix} \exp(\lambda t + \lambda s) & t + s \\ 0 & \exp(\lambda t + \lambda s) \end{pmatrix} = \boldsymbol{\Phi}_3^{t+s}. \end{aligned}$$

$\boldsymbol{\Phi}_3$ therefore is not a flow map. For the functions $\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$, the group property follows directly from the addition theorems

$$\begin{aligned} \sin(t+s) &= \sin(t)\cos(s) + \sin(s)\cos(t), \\ \cos(t+s) &= \cos(t)\cos(s) - \sin(s)\sin(t) \end{aligned}$$

and

$$\begin{aligned} \sinh(t+s) &= \sinh(t)\cosh(s) + \sinh(s)\cosh(t), \\ \cosh(t+s) &= \cosh(t)\cosh(s) + \sinh(s)\sinh(t). \end{aligned}$$

$\boldsymbol{\Phi}_1$ and $\boldsymbol{\Phi}_2$ are flow maps if $\mathbf{y}_i(t) = \boldsymbol{\Phi}_i^t \mathbf{y}_0,\ i = 1, 2$ solves the IVP

$$\dot{\mathbf{y}}_i = \mathbf{f}_i(\mathbf{y}_i), \qquad \mathbf{y}_i(0) = \mathbf{y}_0. \tag{3.2.1}$$

Plugging in yields

$$\mathbf{f}_i(\mathbf{y}_i) = \dot{\boldsymbol{\Phi}}_i^t \mathbf{y}_0 = \dot{\boldsymbol{\Phi}}_i^t (\boldsymbol{\Phi}_i^t)^{-1} \mathbf{y}_i = \dot{\boldsymbol{\Phi}}_i^t \boldsymbol{\Phi}_i^{-t} \mathbf{y}_i.$$

We therefore find that $\boldsymbol{\Phi}_1$ is a flow map to the IVP (3.2.1) with

$$\mathbf{f}_1(\mathbf{y}) = \begin{pmatrix} -\sin(t) & \cos(t) \\ -\cos(t) & -\sin(t) \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \mathbf{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}$$

and $\boldsymbol{\Phi}_2$ is a flow map to the IVP (3.2.1) with

$$\mathbf{f}_2(\mathbf{y}) = \begin{pmatrix} \sinh(t) & \cosh(t) \\ \cosh(t) & \sinh(t) \end{pmatrix} \begin{pmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{pmatrix} \mathbf{y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{y}.$$

Problem 3.3 Discrete Gronwall Lemma

Prove the discrete Gronwall Lemma for constant h:

If the sequence $(\xi_k)_{k\in\mathbb{N}_0}$, $\xi_k \geq 0$ satisfies the inequality

$$\xi_{k+1} \leq Ch^{p+1} + (1+Lh)\xi_k, \quad k \in \mathbb{N}_0, \quad C, h \geq 0,\ L > 0,$$

then

$$\xi_k \leq Ch^p \frac{1}{L}\big(\mathrm{e}^{kLh} - 1\big) + \mathrm{e}^{kLh} \cdot \xi_0, \quad k \in \mathbb{N}_0.$$

HINT: Show, by induction, that

$$\xi_k \leq \frac{Ch^p}{L}\big[(1+Lh)^k - 1\big] + (1+Lh)^k \xi_0$$

and use the convexity of the exponential function.

Solution: *Claim*:

$$\xi_k \leq \frac{Ch^p}{L}\big[(1+Lh)^k - 1\big] + (1+Lh)^k \xi_0$$

For $k = 0$ we get

$$\xi_0 \leq \frac{Ch^p}{L}[1-1] + \xi_0 = \xi_0$$

Induction: Assume, that

$$\xi_k \leq \frac{Ch^p}{L}\big[(1+Lh)^k - 1\big] + (1+Lh)^k \xi_0,$$

then

$$\begin{aligned}
\xi_{k+1} &\leq Ch^{p+1} + (1+Lh)\xi_k \\
&= Ch^{p+1} + (1+Lh)\left(\frac{Ch^p}{L}\big[(1+Lh)^k - 1\big] + (1+Lh)^k \xi_0\right) \\
&= \frac{Ch^p}{L}\big[(1+Lh)^{k+1} - 1\big] + (1+Lh)^{k+1} \cdot \xi_0,
\end{aligned}$$

which proves the claim.

With the convexity of the exponential function, $1 + Lh \leq \mathrm{e}^{Lh}$, the discrete Gronwall Lemma follows

$$\xi_k \leq \frac{Ch^p}{L}\big(\mathrm{e}^{kLh} - 1\big) + \mathrm{e}^{kLh} \cdot \xi_0.$$

Published on 7 March 2016.

To be submitted by 15 March 2016.

References

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

Last modified on April 22, 2016