Problem Sheet 6

Problem 6.1 Invertible Collocation Matrix

Show that, the matrix $\mathfrak{A} \in \mathbb{R}^{s\times s}$ with the entries $(\mathfrak{A})_{ij} = a_{ij}$ is invertible, given that $c_1 > 0$.

HINT: Show that, $\mathfrak{A}\mathbf{z} = 0$ implies $\mathbf{z} = 0$. In order to do this consider the polynomial

$$q(\xi) = \int_0^\xi \sum_{j=1}^s z_j L_j(\tau)\,\mathrm{d}\tau \in \mathcal{P}_s.$$

Problem 6.2 Composition of Runge–Kutta Methods

This exercise investigates the composition of Runge-Kutta methods. It turns out that such a composition is itself always another single step method of Runge-Kutta type:

Let $\hat{\Psi}^{t_0,t_0+h}$ and $\tilde{\Psi}^{t_0,t_0+h}$ be the discrete evolutions for two 2-stage Runge-Kutta single step methods defined by the Butcher-tableaux:

$$\begin{array}{c|cc} 0 & & \\ \widehat{c}_2 & \widehat{a}_{21} & \\ \hline & \widehat{b}_1 & \widehat{b}_2 \end{array} \quad \text{and} \quad \begin{array}{c|cc} 0 & & \\ \widetilde{c}_2 & \widetilde{a}_{21} & \\ \hline & \widetilde{b}_1 & \widetilde{b}_2 \end{array} .$$

Show that their composition

$$\Psi^{t,t+2h} := \tilde{\Psi}^{t+h,t+2h} \circ \hat{\Psi}^{t,t+h}$$

can be interpreted as a discrete evolution of a Runge-Kutta method with step size $2h$, and determine the coefficients of this Runge-Kutta method.

Problem 6.3 Newton's Method as a Single-Step Method

It is a surprising realisation that Newton's method for solving non-linear equation systems can be introduced as a single-step method to solve suitable initial value problems: The damped Newton method for solving $F(\mathbf{y}) = 0$, $F \in \mathcal{C}^2(D, \mathbb{R}^d)$, $D \subset \mathbb{R}^d$ is given as

$$\mathbf{y}_{k+1} = \mathbf{y}_k - s_k(\mathrm{D}F(\mathbf{y}_k))^{-1}F(\mathbf{y}_k), \tag{6.3.1}$$

where $\mathrm{D}F(\mathbf{y}_k)$ is the Jacobian of F and s_k is the damping parameter.

(6.3a) Interpret (6.3.1) as an explicit Euler method for a suitable autonomous ODE. What does the corresponding time grid look like?

(6.3b) Show: If $F(\mathbf{y}^*) = 0$, then $\mathbf{y}^*$ is an attractive fixed point of the differential equation in subproblem (6.3a).

Definition: A fixed point $\mathbf{y}^*$ is called attractive, if there exists an $\epsilon > 0$, such that the solution $t \mapsto \mathbf{y}(t)$ of the initial value problem to every initial value $\mathbf{y}_0$ with $\|\mathbf{y}_0 - \mathbf{y}^*\| < \epsilon$ tends to $\mathbf{y}^*$ for $t \to \infty$.

HINT: Use following theorem:

Let $\mathbf{y}^* \in D$ be a fixed point of the autonomous differential equation $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$ with $\mathbf{f} \in C^1(D, \mathbb{R}^d)$. If the spectral abscissa of the Jacobian of $\mathbf{f}$ in the fixed point $\mathbf{y}^*$ is negative, meaning, if

$$\nu(D\mathbf{f}(\mathbf{y}^*)) := \max_{\lambda \in \sigma(D\mathbf{f}(\mathbf{y}^*))} Re(\lambda) < 0,$$

then $\mathbf{y}^*$ is an attractive fixed point.

Problem 6.4 A MATLAB Function for Implicit Runge-Kutta Methods

Gauss collocation methods are in general implicit methods, therefore a non-linear system of equations must be solved in each step. To this end we use a damped Newton-method (`dampnewton.m`). The construction of collocation methods from [NODE, Sect. 2.2.1] of the lecture gives us a piecewise polynomial approximation $\mathbf{y}_h$ of the solution of the initial value problem, which is given by the collocation conditions on each interval of the time grid, as

$$\mathbf{y}_h(t_k + \tau h) = \mathbf{y}_0 + h \sum_{j=1}^{s} \mathbf{k}_j \int_0^\tau L_j(\xi)\,\mathrm{d}\xi, \quad 0 \le \tau \le 1,$$

with increments $\mathbf{k}_j$.

(6.4a) Extend the function `rkimplss.m`, such that it returns the approximations $\mathbf{y}_h(kh)$ for a given Butcher scheme, the time grid $\{0, h, 2h, \ldots, 1\}$, $h = \frac{1}{N}$ and $k = 0, ..., N$ as well as the values $\mathbf{y}_h$ at the points $\{t_k + \tau_1 h, \ldots, t_k + \tau_l h\}$, $0 < \tau_1 \le \cdots \le \tau_l = 1$.

HINT: Extend `collcoeffs.m`, such that for given $\tau_1, \ldots, \tau_l$, the return value `b` has the components $b_{ji} = \int_0^{\tau_i} L_j(\xi)\,\mathrm{d}\xi$ and then modify `rkimplssm.m`.

(6.4b) Use the modified functions from subproblem (6.4a) and numerically show that an s-step Gauss collocation method on an equidistant timegrid need not adhere to the error estimate

$$\epsilon(h) := \max_{0 \le t \le T} \|\mathbf{y}_h(t) - \mathbf{y}(t)\| = \mathcal{O}(h^{2s}).$$

To do so, complete the template `GaussCollLogRate.m` in which we estimate the convergence rate of a collocation method of order s, with $s = 1, \ldots, 4$, by computing $\epsilon(h)$ on a grid with $N = 2^i$ points, where $i = 2, \ldots, 6$ (i.e. as $h \to 0$).

HINT: As an example, use the scalar logistic differential equation

$$\dot{y} = 10y(1 - y)$$

on $[0, 1]$ with $\mathbf{y}(0) = 0.01$.

(6.4c) Prove that the $\mathbf{y}_h$ computed with the 1-step Gauss collocation method satisfy

$$\max_{0 \leq t \leq T} \|\mathbf{y}_h(t) - \mathbf{y}(t)\| = \mathcal{O}(h^2).$$

Published on 28 March 2016.

To be submitted by 5 April 2016.

References

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 63606.

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

Last modified on April 22, 2016