Numerical Analysis II

Problem Sheet 7

Problem 7.1 Extrapolation of the Implicit Mid-Point Rule.

Taking the implicit mid-point rule as a basis method, we generate another single step method for the autonomous initial value problem $\dot{\mathbf{y}} = f(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$ by extrapolation based on 1 and 2 micro-steps. The right hand side is assumed to be "sufficiently smooth".

(7.1a) Denote by $\mathbf{y}_h$, $\mathbf{y}_{h/2}$ the approximations for $\mathbf{y}(h)$ obtained by (successive) application of the implicit mid-point rule with step sizes h and $h/2$, respectively.

Express the approximate value for $\mathbf{y}(h)$ obtained from the extrapolation method in terms of $\mathbf{y}_{h/2}$ and $\mathbf{y}_h$.

HINT: Use the following theorem concerning the asymptotic expansion of the discretization error of single step methods:

Theorem. Let $\mathbf{y}_N$ be the approximate value of $\mathbf{y}(T)$ obtained by an application of $N := T/h$ steps of a single-step method of order p with step size $h > 0$. Here, $\mathbf{y}(t)$ denotes the solution of the initial value problem $\dot{\mathbf{y}} = f(\mathbf{y})$, $\mathbf{y}(t_0) = \mathbf{y}_0$.

Then there exist smooth functions $\mathbf{e}_i : [t_0, T] \mapsto \mathbb{R}^d$, $i = p, p+1, \ldots, p+k$, with $k \in \mathbb{N}$ determined by regularity of f, and a (for sufficiently small h) uniformly bounded function $(T, h) \mapsto \mathbf{r}_{k+p+1}(T, h)$, such that

$$\mathbf{y}_h - \mathbf{y}(T) = \sum_{l=0}^{k} \mathbf{e}_{l+p}(T) h^{l+p} + \mathbf{r}_{k+p+1}(T, h) h^{k+p+1} \quad \text{for } h \to 0 \ .$$

with $\mathbf{r}_{k+p+1}(T, h) = \mathcal{O}(T - t_0)$ for $T - t_0 \to 0$ uniformly in $h < T$, *where additionally* $\mathbf{e}_l(T) = \mathcal{O}(T - t_0)$ *for* $T \to t_0$.

(7.1b) What is the consistency order of the extrapolation method from subproblem (7.1a)? Justify your answer.

(7.1c) Give the discrete evolution $\boldsymbol{\Psi}^h(y)$ of the implicit mid-point rule for the logistic differential equation

$$\dot{y} = \lambda y(1 - y), \quad \lambda > 0,$$

with initial value $y(0) = y_0 > 0$ in closed form.

HINT: The discrete evolution of the implicit mid-point rule leads to a quadratic equation which can be solved explicitely. Try to solve y_{k+1} as function of y_k out of implicit step function $y_{k+1} = g(y_k, y_{k+1})$.

(7.1d) Which of the two solutions from subproblem (7.1c) is admissible? Justify your answer. HINT: For $f : X \to X$, X be a topological space, define $f^{(n)} = f^{(n-1)}(f(x))$. We say a fixed point x_0 of f is repulsive relative to U, which is an open neighborhood of x_0, if for any open neighborhood V of x_0, there exists a positive integer n_0 such that $f^{(n)}(X - V) \subset X - U$ whenever $n \leqslant n_0$.

(7.1e) Complete the MATLAB template `IMPExtrapLog.m`, which is supposed to implement the extrapolated mid-point rule for the initial value problem from subproblem (7.1c).

Plot both your numerical result for the time interval $[0, 1]$ for the parameter value $\lambda = 10$ and initial value $y(0) = 0.2$ as well as the exact solution

$$y(t) = \frac{1}{1 + (y_0^{-1} - 1)e^{-\lambda t}}$$

in a figure and save the plot in `logistic.eps`.

(7.1f) Complete the MATLAB template `IMPExtrapLogConv.m`, to determine the convergence order of the extrapolation method for the logistic differential equation empirically. For this, use the initial value problem from subproblem (7.1e) and the step size $h = 1/2^n$, $n = 4, 5, \dots, 9$. Save the relevant plot in the file `conv.eps`.

Problem 7.2 Extrapolation methods as Runge-Kutta methods

Extrapolation methods based on explicit methods can be written as explicit Runge–Kutta methods. Here, we will choose the explicit Euler method as our base method. Perform an extrapolation step with step size $h_1 = 2\,h_2$. Then, write the extrapolated method as a Runge–Kutta method (Butcher-tableau) and determine its order of consistency.

HINT: Use the Aitken-Neville algorithm [NUMODE, Eq. (2.4.5)],[NUMODE, Eq. (2.4.6)],[NUMODE, Sect. 2.4.2].

Problem 7.3 Low Storage Runge-Kutta Method

A low storage Runge-Kutta method of order n is an algorithm for the solution of the autonomous IVP $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$, $\mathbf{y}(0) = \mathbf{y}_0$ of the form

$\mathbf{x}_0 = \mathbf{y}_0$
$\mathbf{q}_0 = 0$
for $j = 1$ **to** n **do**
  $\mathbf{q}_j = \alpha_j \mathbf{q}_{j-1} + h\mathbf{f}(\mathbf{x}_{j-1})$
  $\mathbf{x}_j = \mathbf{x}_{j-1} + \beta_j \mathbf{q}_j$
end for
$\mathbf{y}_1 = \mathbf{x}_n$

The algorithm is described by the real coefficients $(\alpha_j)_{j=1}^n$, $(\beta_j)_{j=1}^n$, where $\alpha_1 = 0$.

(7.3a) Show that the 2-stage explicit autonomization invariant Runge-Kutta methods of order 2 form a one-parameter family. Determine the corresponding Butcher-Tableau as a function of this parameter.

HINT: Prove that all coefficients of Butcher tableau are functions of the same single parameter.

(7.3b) Show that the stability function of any 2-stage explicit autonomization invariant Runge-Kutta method of order 2 satisfies

$$S(z) = 1 + z + \frac{z^2}{2}.$$

(7.3c) Show that every 2-stage explicit autonomization invariant Runge-Kutta method of order 2 is a 2-stage low storage Runge-Kutta method.

(7.3d) A n-stage low storage Runge-Kutta autonomization invariant method is also a Runge-Kutta method in the sense of [NODE, Def. 2.2.2]. Prove this fact for the special case of a 3-stage low storage Runge-Kutta method by determination of the Butcher-Tableau of the corresponding Runge-Kutta method.

Published on 11 April 2016.

To be submitted by 19 April 2016.

References

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 52913.

Last modified on May 24, 2016