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Numerical Analysis II

ETH Zürich
D-MATH

Problem Sheet 9

Problem 9.1 Stability Domain of a Rational Single Step Method

Consider the rational function

$$R(z) = \frac{2 - z^2}{2(1-z)}.$$

(9.1a) Determine the maximal $p \in \mathbb{N}$ such that

$$|\exp(z) - R(z)| = \mathcal{O}(|z|^{p+1}) \qquad \text{for } z \to 0.$$

HINT: Compute the first three derivatives of $R(z)$ and use them to compare the Taylor series of $\exp(z)$ and $R(z)$ around the point 0.

(9.1b) Consider $R(z)$ as a stability function of a Runge-Kutta single step method and plot its stability domain in MATLAB by completing the template `StabilityRegion.m`.

(9.1c) Show that a Runge-Kutta method with stability function $R(z)$ is of convergence order 2 when applied to linear ODEs, that is, to problems of the form $\dot{y} = \lambda y,\ y(0) = y_0$.

(9.1d) Write down (in detail) the discrete evolution of the single step method (whose stability function is $R(z)$), when applied to the autonomous linear differential equation

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}, \qquad \mathbf{A} \in \mathbb{R}^{d\times d}. \tag{9.1.1}$$

(9.1e) Implement the method (in MATLAB) for the approximate solution of (9.1.1) by completing the template `RationalSSM.m` to solve the initial value problem

$$\dot{\mathbf{y}} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

for $t \in [0, 10]$ with the values

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
α	-2	-2	-2	1.5	1.5	1.5
β	-1	-2	-2	0	0	0
h	1	1	0.5	0.5	1	1.5

where h is the step size. Plot your results and compare them with the exact solution. Explain the behaviour of the method with the help of the stability domain of $R(z)$.

Problem 9.2 Stability-Induced Bound on Step Size.

The stability domains of explicit Runge-Kutta methods are necessarily bounded, see [NODE, Lemma. 3.1.10]. This leads to a stability-induced bound on the step size in the vicinity of asymptotically stable fixed points.

Consider the logistic differential equation $\dot{y} = \lambda y(1-y)$, $\lambda > 0$, with asymptotically stable fixed point $y = 1$.

Determine numerically an optimal bound on the step size (in dependence of λ) for the classical Runge-Kutta method and the embedded method of order 4(5) of Merson (see [NUMODE, Ex. 2.6.14]) in such a way, that the stability of the fixed point $y = 1$ is only just preserved by the discretization.

Complete the template `stabfn.m`, `stpRestrict.m`. Test your bound in a numerical experiment and plot with `stabdomRK.m`.

HINT: [NODE, Thm. 3.2.8].

Problem 9.3 Diagonalizable matrices are dense in $\mathbb{C}^{d\times d}$

Prove that the set of all diagonalizable $d \times d$ complex matrices is dense in $\mathbb{C}^{d\times d}$.

HINT: Use the Schur decomposition and the fact that a matrix with pairwise distinct eigenvalues is diagonalizable.

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References

[NODE] Lecture Notes for the course "Numerical Methods for Ordinary Differential Equations".

[NUMODE] Lecture Slides for the course "Numerical Methods for Ordinary Differential Equations", SVN revision # 52913.

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