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https://www2.math.ethz.ch/education/bachelor/lectures/fs2016/math/wr_s

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Exercise Series 2

Q1. The birthday paradox Take an urn with N balls numerated from $\{1, .., N\}$. Perform the experiment of extracting balls with replacement.

(a) Let A_n := "The first n balls extracted are different". Calculate $\mathbb{P}(A_n)$ (use a Laplace model).

(b) Prove the following inequalities:

$$1 - \frac{n(n-1)}{2N} \leq \mathbb{P}(A_n) \leq \exp\left(-\frac{n(n-1)}{2N}\right).$$

(c) Calculate $n_{\min} = \inf\{n \in \mathbb{N} : \mathbb{P}(A_n) < \frac{1}{2}\}$ for $N = 365$. Relate this problem with the Birthday Problem: " Find the probability that, in a group of n people, there is at least one pair who have the same birthday".

Q2. We are interested in studying the probability of success of a student at an entrance exam to two departments of a university. Consider the following events

$$A = \{\text{The student is man}\},$$
$$A^c = \{\text{The student is woman}\},$$
$$B = \{\text{The student applied for department I}\},$$
$$B^c = \{\text{The student applied for department II}\},$$
$$C = \{\text{The student was accepted}\},$$
$$C^c = \{\text{The student wasn't accepted}\}.$$

We assume that we have the following probabilities (Berkeley 1973):

$\mathbb{P}(A) = 0.73,$
$\mathbb{P}(B \mid A) = 0.69,\ \ \mathbb{P}(B \mid A^c) = 0.24,$
$\mathbb{P}(C \mid A \cap B) = 0.62,\ \ \mathbb{P}(C \mid A^c \cap B) = 0.82,\ \ \mathbb{P}(C \mid A \cap B^c) = 0.06,\ \ \mathbb{P}(C \mid A^c \cap B^c) = 0.07.$

(a) Draw a tree describing the situation with the probabilities associated.

(b) Taking in to consideration the following probabilities $\mathbb{P}[C|A \cap B] = 0.62$, $\mathbb{P}[C|A^c \cap B] = 0.82$, $\mathbb{P}[C|A \cap B^c] = 0.06$, $\mathbb{P}[C|A^c \cap B^c] = 0.07$. With this information, do you think that in this examination women are disadvantaged?

(c) Compute $\mathbb{P}(C \mid A)$ and $\mathbb{P}(C \mid A^c)$. Does this coincide with your answer of b)?

Q3. POSTERIOR PROBABILIES Suppose that a box contains three coins and that for each coin there is a different probability that a head will be obtained when the coin is tossed. Let p_i denote the probability of a head when the ith coin is tossed ($i = 1, \dots, 3$), and suppose that $p_1 = 1/4$, $p_2 = 1/2$, $p_3 = 3/4$.

(a) Suppose that one coin is selected uniformly at random from the box and when it is tossed once, a head is obtained. What is the posterior probability that the ith coin was selected?

(b) If the same coin were tossed again, what would be the probability of obtaining another head?

(c) Prove the CONDITIONAL BAYES' THEOREM: Let $(A_i)_{i=1 \cdots k}$ be a partition of Ω, and B, C are events in Ω,

$$\mathbb{P}(A_i|B \cap C) = \frac{\mathbb{P}(A_i|B)\mathbb{P}(C|A_i \cap B)}{\sum_{j=1}^{k} \mathbb{P}(A_j|B)\mathbb{P}(C|A_j \cap B)}.$$

(d) If the same coin gives another head at the second toss, what is the posterior probability that the ith coin was selected?

(e) Assume that it is always the same coin tossed, and we get always head. What is the recurrence relation of the posterior probability after n tosses that the ith coin was selected?

Q4. INTRODUCTION TO BAYESIAN STATISTICS We have m urns with red and white balls inside. The urn $i \in \{1, ..m\}$ has $2i - 1$ red balls and $2m - 2i + 1$ white ones. We randomly select an urn and extract with replacement n times. Define:

$$X_j := \begin{cases} 1 & \text{If the } j\text{-th ball is red,} \\ 0 & \text{If the } j\text{-th ball is white.} \end{cases}$$

We are interested in the following problem " Given that you see $(X_j)_{j=1}^{n}$, can you say from which urn the balls where taken?"

(a) Compute $\mathbb{P}(X_1 = x_1, ..., X_n = x_n)$ for $x_i \in \{0, 1\}$. Are $X_1, .., X_n$ independent?

(b) Compute the following probability:

$$\mathbb{P}\left(\text{The urn chosen is } i \mid X_1 = x_1, ..., X_n = x_n\right).$$

Show that this only depends on the number or red balls, i.e., $k = \sum_{i=1}^{n} x_i$.

(c) Compute $\mathbb{P}\left(\text{The urn chosen is } i \mid X_1 = x_1, ..., X_n = x_n\right)$ for $m = 3$ and $n = 3$.

Have fun!