

Exercise Series 3

Q1. In a clinical trial with two treatment groups, the probability of success in one treatment group is 0.5, and the probability of success in the other is 0.6. Suppose that there are five patients in each group. Assume that the outcomes of all patients are independent. Calculate the probability that the first group will have at least as many successes as the second group.

Q2. Recall that a *Poisson process* with rate λ per unit time is a process that satisfies two properties:

- (a) The number of arrivals in every fixed interval of time of length t has the Poisson distribution with mean λt .
- (b) The numbers of arrivals in every collection of disjoint time intervals are independent.

Suppose that the arrival time of clients to store A is a Poisson process with rate 1 per hour.

- (a) What is the probability of “the first client comes later than t hours”?
- (b) What is the probability of “more than two clients come during the first hour”?
- (c) Fix a time $t > 0$, what is the probability of “a client comes at some exact time t ”? Does it mean that nobody comes at any time? Is that contradictory?
- (d) Suppose that the arrival time of clients to another store B is a Poisson process with rate μ per hour which is independent to the store A , now assumed to have rate λ . What are arrival times of clients to both store (if we forget about in which store they are)?

Q3. BOREL CANTELLI

- (a) Construct a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a series of measurable sets $(A_n)_{n \in \mathbb{N}}$ with $\sum_{n \in \mathbb{N}} \mathbb{P}(A_n) = \infty$ and $\mathbb{P}(\bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k) = 0$.
- (b) Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Take $(U_n)_{n \in \mathbb{N}}$ a series of uniform independent random variables on $(0, 1)$, i.e., for $0 \leq x \leq 1$, $\mathbb{P}(U_n \in [0, x]) = x$.

i. Show that:

$$\mathbb{P}((\exists \alpha > 1) \liminf n^\alpha U_n \in \mathbb{R}) = 0.$$

Hint: It may be useful to define, for $\alpha > 1$ $A_n^\alpha := \{U_n < n^{-\alpha}\}$. Do not forget that the countable union of sets of probability 0 has probability 0.

ii. Prove that:

$$\mathbb{P}(\liminf nU_n \in \mathbb{R}) > 0.$$

- Q4.** THE BERTRAND'S PARADOX Consider an equilateral triangle inscribed in a circle of radius 1. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?. For solving this try the following probability models:
- (a) The "random endpoints" method: Choose two uniform random points on the circumference of the circle and draw the chord joining them, i.e., let $U, V \sim U(0, 1)$, define $X = e^{U2\pi i}, Y = e^{V2\pi i}$ and take the chord connecting X and Y .
 - (b) The "random radius" method: Choose a radius of the circle, choose a uniform point on the radius and construct the chord through this point and perpendicular to the radius, i.e., choose a radius and choose $r \sim U(0, 1)$, take the point on the radius that is at distance r from the center and the chord will be the only chord perpendicular to this point.
 - (c) The "random midpoint" method: Choose a point uniformly anywhere within the circle and construct a chord with the chosen point as its midpoint, i.e., take $(x, y) \sim U(B(0, 1))$ and take the chord whose midpoint is (x, y) .
 - (d) Is this a contradiction?