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Exercise Series 4

Q1. **(a)** Take $p \in [0,1]$ and $n \in \mathbb{N}\backslash\{0\}$. We say that $X \sim Bin(n,p)$ if the distribution of X is

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k \in \{0,1,...,n\}.$$

Show that this is indeed a probability distribution using 2 different methods:

i. Calculating $\sum_k \mathbb{P}(X=k)$.

ii. Representing this probability in terms of the box model with replacement.

(b) Take $K, n \in \mathbb{N}$ and $N \in \mathbb{N}\backslash\{0\}$ with $K, n \leq N$. We say that a random variable $X \sim Hyp(N,K,n)$ if its distribution is given by

$$\mathbb{P}(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} \qquad k \in \{\max\{0, n+K-N\}, .., \min\{n, K\}\}$$

Show that this is indeed a probability distribution using 2 different methods:

i. Calculating $\sum_k \mathbb{P}(X=k)$.
Hint: Calculate $(1+x)^n$ in two different ways and identify the terms.

ii. Representing this probability in the box model without replacement.

Q2. Let X_1 and X_2 follow a normal distribution with mean μ_i and variance σ_i^2. Prove that if X_1 is independent of X_2 then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

Q3. Let X be a standard normal random variable.

(a) Prove that if we take $Y := X^2$, then $f_Y(y) = ce^{-y/2}y^{-1/2}\mathbf{1}_{y\geq 0}$ (We say that Y is distributed according to a χ-squared with one degree of freedom).

(b) If Y_1 and Y_2 are two independent copies of Y, prove that $f_{Y_1+Y_2}(x) = c_2 e^{-x/2}\mathbf{1}_{x\geq 0}$. What is the name of this distribution.

(c) With the help of induction prove that $\sum_{i=1}^n Y_i$, where $(Y_i)_{i=1}^n$ are independent copies of Y, has as a density function

$$f_{\sum_{i=1}^n Y_i}(x) = c_n x^{\frac{n}{2}-1} e^{-\frac{x}{2}} \mathbf{1}_{x\geq 0}.$$

This is call a χ-squared distribution with n degrees of freedom.

Q4. Memorylessness of exponential random variable. We say that a random variable X has an exponential distribution of parameter λ (write it $\mathcal{E}(\lambda)$) if for all $t \geq 0$:

$$\mathbb{P}(X \geq t) = e^{-\lambda t}.$$

(a) Find the density function (with respect to the Lebesgue Measure) of an exponential random variable.

(b) Show that if $X_1 \sim \mathcal{E}(\lambda_1)$, $X_2 \sim \mathcal{E}(\lambda_2)$ and $X_1 \perp X_2$, then $\min\{X_1, X_2\} \sim \mathcal{E}(\lambda_1 + \lambda_2)$.

(c) Show that

$$\mathbb{P}\left(X \geq t + h \mid X \geq h\right) = \mathbb{P}(X \geq t).$$

This property is called memorylessness. We want to prove that the only random variable that has the memorylessness property is the exponential random variable. Suppose that $Y : \Omega \mapsto \mathbb{R}^+$ has the memorylessness property, i.e.,

$$\mathbb{P}\left(Y \geq t + h \mid Y \geq h\right) = \mathbb{P}(X \geq t).$$

(d) Define $G(t) := \mathbb{P}(Y \geq t)$ and prove that $G(t + h) = G(t)G(h)$.

(e) Prove that for all $m, n \in \mathbb{N}$, $G\left(\frac{m}{n}\right) = G(1)^{\frac{m}{n}}$.

(f) Using the monotone property of G prove that for all $t \geq 0$, $G(t) = G(1)^t$. Conclude that Y has an exponential distribution and make explicit the parameter.