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# Exercise Series 5

**Q1.** **(a)** Take $p \in [0,1]$ and $n \in \mathbb{N}\backslash\{0\}$. We say that $X \sim Bin(n,p)$ if the distribution of $X$ is

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k \in \{0, 1, ..., n\}.$$

Show that this is indeed a probability distribution using 2 different methods:

i. Calculating $\sum_k \mathbb{P}(X = k)$.

ii. Representing this probability in terms of the box model with replacement.

Calculate the expected value of $X$ using 2 different methods (the one listed above).

**(b)** Take $K, n \in \mathbb{N}$ and $N \in \mathbb{N}\backslash\{0\}$ with $K, n \leq N$. We say that a random variable $X \sim Hyp(N, K, n)$ if its distribution is given by

$$\mathbb{P}(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} \qquad k \in \{\max\{0, n+K-N\}, .., \min\{n, K\}\}$$

Show that this is indeed a probability distribution using 2 different methods:

i. Calculating $\sum_k \mathbb{P}(X = k)$.
**Hint:** Calculate $(1+x)^n$ in two different ways and identify the terms.

ii. Representing this probability in the box model without replacement.

Calculate the expectation using both methods.

**Q2.** **(a)** Let $X \sim \mathcal{E}(\lambda)$ be the exponential random variable with parameter $\lambda$. Compute the $\alpha$-quantile for all $\alpha \in (0,1)$.

**(b)** Let $U$ be uniformly distributed random variable in $\{1, 2, \cdots, N\}$. Determine the CDF of $U$, compute the $\alpha$-quantile for all $\alpha \in (0,1)$. For which $\alpha$ the quantile is uniquely determined?

**Q3.** Let $X$ and $Y$ be random variables with joint density distribution given by

$$f_{X,Y}(x,y) = e^{-x^2 y} \mathbf{1}_{x \geq 1} \mathbf{1}_{y \geq 0}$$

**(a)** Why is this a probability measure?

**(b)** What is the density function of $X$.

**(c)** Compute $\mathbb{P}(Y \leq 1/X^2)$.

**Q4.** Let $X$ and $Y$ be two independent exponential random variables with parameter $\lambda$. Let $a > 0$.

**(a)** What is the joint density of the couple of random variables $(X, X+Y)$?

**(b)** Let $b \leq a$, what is the probability of $X \leq b$ conditioned on the event

$$B := \{X \leq a \text{ and } X + Y \geq a\}.$$

**(c)** What is the conditional density of $X$ given the event $B$?

**Q5.** **(a)** Take $X$ a random variable. Prove that for all $\lambda \geq 0$

$$\mathbb{P}(X \geq t) \leq \exp(-\lambda t)\mathbb{E}\left(\exp(\lambda X)\right).$$

**(b)** Define $\phi_X(\lambda) := \ln\left(\mathbb{E}\left(e^{\lambda X}\right)\right)$. Prove that $\phi(\lambda) \geq \lambda \mathbb{E}\left(X\right)$.

**(c)** Prove that

$$\mathbb{P}(X \geq t) \leq \exp(-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}).$$

**(d)** If $X \sim N(0, \sigma)$, calculate $\phi_X(\lambda)$.

**(e)** Prove that if $X$ is a positive random variable

$$\mathbb{E}\left(X\right) = \int_0^\infty \mathbb{P}(X \geq t)dt$$

**(f)** Show that if $X \sim N(0, \sigma)$ and $Y$ is a random variable such that $\phi_Y(\lambda) \leq \phi_X(\lambda)$, then

$$\mathbb{E}(Y^2) \leq 4\sigma^2.$$