Lecturer: Prof. Dr. Mete SONER
Coordinator: Yilin WANG

ETH Zürich

Exercise Series 6

Q1. Suppose that internet users access a particular Web site according to a Poisson process with rate Λ per hour, but Λ is unknown. The Web site maintainer believes that Λ has a continuous distribution with probability density function:

$$f(\lambda) = \begin{cases} 2e^{-2\lambda} & \text{for } \lambda > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Let X be the number of users who access the Web site during a one-hour period. If $X = 1$ is observed, find the conditional p.d.f. of Λ given $X = 1$.

Q2. Let X be a real-valued random variable. We define the characteristic function of X by

$$\varphi_X : \mathbb{R} \to \mathbb{C}$$
$$t \mapsto \varphi_X(t) := \mathbb{E}[e^{itX}] = \int e^{itx} \mu(dx),$$

in which μ is the distribution of X on $\mathbb{R}$. It represents an important analytic tool, that the distribution of a random variable is uniquely determined (characterized) by the characteristic function. Show the following features:

(a)
- $\varphi_X(0) = 1$,
- $|\varphi_X(t)| \leq 1$,
- φ_X is continuous, and
- $\varphi_{aX+b}(t) = e^{itb}\varphi_X(at)$ for all $a, b \in \mathbb{R}$.

(b) Show that if the n-th moment of X exists, i.e. $\mathbb{E}[|X|^n] < \infty$, then φ_X is n times differentiable, and

$$\varphi_X^{(k)}(t) = i^k \mathbb{E}[X^k e^{itX}], \quad \text{for all } k \leq n,$$

(in particular $\varphi_X^{(k)}(0) = i^k \mathbb{E}[X^k]$).

Hint: one can use the inequality $|\frac{e^{i\alpha}-1}{\alpha}| \leq 1$, $(\alpha \in \mathbb{R})$.

(c) Compute the characteristic function for the standard normal distribution $\mathcal{N}(0, 1)$, then for $\mathcal{N}(\mu, \sigma^2)$.

(d) Let X and Y be two independent random variables, defined on the same probability space. What is the characteristic function of $X + Y$?

Let $X_1, X_2, \cdots, X_n$ be random variables defined on the same probability space. $(X_1, X_2, \cdots, X_n)$ is said to be a *Gaussian vector* if all $\mathbb{R}$-linear combinations of X_i are centered Gaussian random variable.

Q3. Let X and Y two independent standard normal random variables ($\mathcal{N}(0,1)$). Define the random variable

$$Z := \begin{cases} X & \text{if } Y \geq 0, \\ -X & \text{if } Y < 0. \end{cases}$$

(a) Compute the distribution of Z.

(b) Compute the correlation between X and Z.

(c) Compute $\mathbb{P}(X + Z = 0)$.

(d) Does (X, Z) follow a multivariate normal distribution (in other words, is (X, Z) a Gaussian vector)?

Q4. Assume that $X := (X_1, X_2, \cdots, X_n)$ is a Gaussian vector, K_X the covariance matrix of X, which is defined by

$$K_X(i,j) = Cov(X_i, X_j).$$

(a) Let $\alpha_1, \cdots, \alpha_n$ be n real numbers, what is the law of $\sum_{i=1}^{n} \alpha_i X_i$ in terms of K_X?

(b) What can you say about K_X?

(c) If $K_X(1,2) = 0$, show that X_1 and X_2 are independent. Is that true if we don't assume that (X_1, X_2) is a Gaussian vector?

Hint: The characteristic function of the pair of random variables $X := (X_1, X_2)$, which is defined as

$$\varphi_X : \mathbb{R}^2 \to \mathbb{C}$$
$$t = (a,b) \mapsto \varphi_X(t) := \mathbb{E}[e^{it \cdot X}] = \mathbb{E}[e^{i(aX_1 + bX_2)}],$$

characterizes also the joint law of (X_1, X_2).