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Exercise Series 8

Q1. If $\log(X)$ has the normal distribution with mean μ and variance σ^2, we say that X has the *lognormal distribution* with parameters μ and σ^2.

A popular model for the change in the price of a stock over a period of time of length u is to say that the price after time u is $S_u = S_0 Z_u$, where Z_u has the lognormal distribution with parameter μu and $\sigma^2 u$. In this formula, S_0 is the present price of the stock, and σ is called the *volatility* of the stock price.

(a) What is the expected value of S_1?

(b) Find the distribution of $1/S_1$.

(c) What is the expected value of $1/S_1$?

(d) What are k-th moments of S_1, for $k = 1, 2, \ldots$?

Q2. Suppose that Z has the standard normal distribution, V has the χ-squared distribution with n degrees of freedom, and that Z and V are independent. Let

$$T = \frac{Z}{\sqrt{V/n}}.$$

You will show that T has p.d.f. given by

$$f(t) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \qquad t \in \mathbb{R}.$$

Recall that we have seen in Series 4, that the p.d.f. of a χ-squared with n degrees of freedom is, for some $c_n \in \mathbb{R}$:

$$f_V(x) = c_n x^{n/2-1} e^{-x/2} \mathbf{1}_{\{x \geq 0\}}.$$

(a) Find the joint p.d.f. of (T, V).

(b) Show first that the conditional distribution of T given $V = v$ is normal with mean 0 and variance $\frac{n}{v}$.

(c) Compute c_n.

(d) Find the p.d.f. of T.

Q3. We would like to compute

$$A := \int_{-3}^{1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

using Monte-Carlo method:

Table 1: Data for Q4.(b)

i	x_i	y_i
1	1.9	0.7
2	0.8	-1.0
3	1.1	-0.2
4	0.1	-1.2
5	-0.1	-0.1
6	4.4	3.4
7	4.6	0.0
8	1.6	0.8
9	5.5	3.7
10	3.4	2.0

(a) Express A under the form $\mathbb{E}(f(U))$, where U is a standard Gaussian random variable, and f an appropriate function.

(b) Take $(U_i)_{i\in\mathbb{N}}$ an i.i.d. family having the same law as U. Set

$$S_n := \frac{1}{n}\sum_{i=1}^{n} f(U_i).$$

What is the distribution of $S_n - A$?

(c) Compute $\mathbb{E}(S_n - A)$ and show that $Var(S_n - A) = (A - A^2)/n$.

(d) Show that for any $x > 0$, $\mathbb{P}(|S_n - A| \geq x) \leq 1/nx^2$, thus converges to 0 when $n \to \infty$.

(e) Which theorem can you apply to get directly the above convergence?

(f) Calculate A using your favorite software.

Q4. *Fitting a polynomial by Methode of Least Squares* Suppose now that instead of simply fitting a straight line to n plotted points, we wish to fit a polynomial of degree k ($k \geq 2$). such a polynomial will have the following form:

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_k x^k.$$

The method of least squares specifies that the constants $\beta_0, \cdots, \beta_k$ should be chosen that the sum

$$Q(\beta_0, \cdots, \beta_k) = \sum_{i=1}^{n}[y_i - (\beta_0 + \beta_1 x_i + \cdots + \beta_k x_i^k)]^2$$

of the squares of the vertical deviations of the points from the curve is a minimum.

(a) Which equation system should a minimizer $\hat{\beta}_0, \cdots, \hat{\beta}_k$ satisfy?

(b) Fit a parabola (polynomial of degree 2) to the 10 points given in the table.