**Lecturer:** Prof. Dr. Mete SONER
**Coordinator:** Yilin WANG

ETH Zürich

# Exercise Series 9

**Q1.** Let $\alpha, \beta > 0$, the p.d.f. of a beta distribution with parameters $\alpha$ and $\beta$ is

$$f(x|\alpha,\beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $X_1, \cdots, X_n$ form a random sample from the Bernoulli distribution with parameter $\theta$, which is unknown ($0 < \theta < 1$). Suppose also that the prior distribution of $\theta$ is the beta distribution with parameters $\alpha > 0$ and $\beta > 0$. Show that the posterior distribution of $\theta$ given that $X_i = x_i (i = 1, \ldots, n)$ is the beta distribution with parameters $\alpha + \sum_{i=1}^n x_i$ and $\beta + n - \sum_{i=1}^n x_i$.

In particular the family of beta distributions is a conjugate family of prior distributions for samples from a Bernoulli distribution. If the prior distribution of $\theta$ is a beta distribution, then the posterior distribution at each stage of sampling will also be a beta distribution, regardless of the observed values in the sample.

**Q2.** Let $\xi(\theta)$ be defined as follows: for constants $\alpha > 0$ and $\beta > 0$:

$$\xi(\theta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta} & \text{for } \theta > 0, \\ 0 & \text{for } \theta \le 0. \end{cases}$$

A distribution with this p.d.f. is called an *inverse gamma distribution*.

**(a)** Verify that $\xi(\theta)$ is actually a p.d.f..

**(b)** Consider the family of probability distributions that can be represented by a p.d.f. $\xi(\theta)$ having the given form for all possible pairs of constants $\alpha > 0$ and $\beta > 0$. Show that this family is a conjugate family of prior distributions for samples from a normal distribution with a known value of the mean $\mu$ and an unknown value of the variance $\theta$.

**Q3.** Suppose that the number of defects in a 1200-foot roll of magnetic recording tape has a Poisson distribution for which the value of the mean $\theta$ is unknown, and the prior distribution of $\theta$ is the gamma distribution with parameters $\alpha = 3$ and $\beta = 1$. When five rolls of this tape are selected at random and inspected, the numbers of defects found on the rolls are $2, 2, 6, 0$ and 3.

**(a)** What is the posterior distribution of $\theta$?

**(b)** If the squared error loss function is used, what is the Bayes estimate of $\theta$?

**Q4.** Let $c > 0$ and consider the loss function

$$L(\theta, a) = \begin{cases} c|\theta - a| & \text{if } \theta < a, \\ |\theta - a| & \text{if } \theta \geq a. \end{cases}$$

Assume that $\theta$ has a continuous distribution.

**(a)** Let $a \leq q$ be two real numbers, show that

$$\mathbb{E}[L(\theta, a) - L(\theta, q)] \geq (q - a)[\mathbb{P}(\theta \geq q) - c\mathbb{P}(\theta \leq q)],$$

and

$$\mathbb{E}[L(\theta, a) - L(\theta, q)] \leq (q - a)[\mathbb{P}(a \leq \theta) - c\mathbb{P}(\theta \leq a)].$$

**(b)** Prove that a Bayes estimator of $\theta$ will be any $1/(1 + c)$ quantile of the posterior distribution of $\theta$.