Lecturer: Prof. Dr. Mete SONER
Coordinator: Yilin WANG

ETH Zürich

Solution Series 1

Q1. We throw simultaneously two dices, one green and one red. Consider the following events:

- $W_1 :=$ Neither of the dices has a result greater than 2.
- $W_2 :=$ The green and the red one have the same number on them.
- $W_3 :=$ The number on the green is 3 times the number on the red.
- $W_4 :=$ The number on the red is by one greater than the number on the green one.
- $W_5 :=$ The number of the green one is greater or equal than the number on the red one.

(a) Write a suitable space Ω where all of these events can live.

(b) Describe W_i as a subsets of Ω.

(c) If you were colorblind (you cannot differentiate green and red). How does the sample space Ω change? Which W_i can live in this space?

Solution

(a) $\Omega = \{1,2,3,4,5,6\}^2$. The first coordinate will represent the green dice and the second one the red dice.

(b)
- $W_1 := \{(x,y) \in \Omega : x \leq 2, y \leq 2\} = \{1,2\}^2$.
- $W_2 := \{(x,y) \in \Omega : x = y\} = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$.
- $W_3 := \{(x,y) \in \Omega : x = 3y\} = \{(3,1),(6,2)\}$.
- $W_4 := \{(x,y) \in \Omega : x+1 = y\} = \{(1,2),(2,3),(3,4),(4,5),(5,6)\}$.
- $W_5 := \{(x,y) \in \Omega : x \geq y\}$.

(c) We can define the equivalence relation $\sim$ as

$$(x,y) \sim (z,w) \Leftrightarrow \{x,y\} = \{z,w\},$$

then $\tilde{\Omega} = \Omega / \sim = \{\{x,y\} : x,y \in \Omega\}$. The W_i that will survive are those so that if $(x,y) \in W_i$ then $(y,x) \in W_i$. This happens only for W_1, W_2.

Q2. You have an urn with $4k$ balls each one numerated with a different number in $\{1,...,4k\}$. At time j you take out one ball, look at its number and put it back, you repeat this experiments n times. Define

- $A_j :=$ The number taken out in the j-th time is bigger than $2k$.

- $B_j :=$ The number taken out in the j-th time is even.

(a) Write in terms of $(A_j)_{j=1}^n$ and $(B_j)_{j=1}^n$ the following events

i. $A :=$ Between 1 and n there was never a number bigger than $2k$.

ii. $B :=$ Between 1 and n there was at least one even number.

iii. $C :=$ The amount of balls bigger than $2k$ is bigger or equal than the amount of even balls.

(b) Describe in words the following events

i. $\left(\bigcup_{j=1}^{n}(A_j)^c\right)^c$.

ii. $\bigcup_{j=1}^{n-2}(A_j \cap A_{j+1} \cap B_{j+2})$.

iii. $\bigcup_{m=1}^n \bigcap_{j=m}^n (A_j \cap B_j)$.

Solution

(a) i. $A = \bigcap_{j=1}^n A_j^c$.

ii. $B = \bigcup_{j=1}^n B_j$.

iii. $C = \bigcup_{\substack{C,D\subseteq\{1,\dots,n\}\\ |C|=|D|}} \left(\bigcap_{j\in C} A_j \cap \bigcap_{j\in D^c} B_j^c\right)$.

(b) i. $\left(\bigcup_{j=1}^{n}(A_j)^c\right)^c = \bigcap_{j=1}^n A_j$: All the number taken are bigger than $2k$.

ii. $\bigcup_{j=1}^{N-2}(A_j \cap A_{j+1} \cap B_{j+2})$: There exists one moment where in two consecutive drawings we got a number bigger than $2k$ and in the following extraction we got an even number.

iii. $\bigcup_n^N \bigcap_{j=n}^N A_j \cap B_j$:There is a moment after we only extract number which are even and bigger than $2k$.

Q3. Let $(A_j)_{j=1}^n$ be events, $A_j \subset \Omega$ and for every event A, let $\mathbf{1}_A$ denote the indicator function of A, which is a function from Ω to $\{0,1\}$ such that $\mathbf{1}_A(w) = 1$ if $w \in A$, and $\mathbf{1}_A(w) = 0$ otherwise.

(a) Show that:

$$\mathbf{1}_{\bigcup_{j=1}^n A_j} = 1 - \prod_{j=1}^n (1 - \mathbf{1}_{A_j}),$$

use it to prove that:

$$\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] = \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le i_1<\ldots<i_k\le n} \mathbb{P}\left[\bigcap_{j=1}^{k} A_{i_j}\right].$$

(b) Using induction prove the following statements:

$$\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] \le \sum_{j=1}^{n}\mathbb{P}[A_j] - \sum_{j=1}^{n-1}\mathbb{P}\left[A_j\cap A_{j+1}\right],$$

$$\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] \ge \sum_{j=1}^{n}\mathbb{P}[A_j] - \sum_{i,j=1,i<j}^{n}\mathbb{P}\left[A_j\cap A_{i}\right].$$

Solution

(a) It's clear that

$$\begin{aligned}\mathbf{1}_{\bigcup_{j=1}^n A_j} &= 1 - \mathbf{1}_{\bigcap_{j=1}^n A_j^c}\\ &= 1 - \prod_{j=1}^{n}\mathbf{1}_{A_j^c}\\ &= 1 - \prod_{j=1}^{n}\left(1-\mathbf{1}_{A_j}\right),\end{aligned}$$

Then we can just use the product formula to have:

$$\prod_{j=1}^{n}\left(1-\mathbf{1}_{A_j}\right) = 1 + \sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\prod_{j=1}^{k} -\mathbf{1}_{A_{i_k}}.$$

Taking expectations at both sides

$$\begin{aligned}\mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] &= \mathbb{E}\left[\mathbf{1}_{\bigcup_{j=1}^n A_j}\right]\\ &= -\mathbb{E}\left[\sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\prod_{j=1}^{k} -\mathbf{1}_{A_{i_k}}\right]\\ &= \sum_{k=1}^{n}(-1)^{k+1}\sum_{1\le i_1<\ldots<i_k\le n}\mathbb{P}\left[A_{i_1}\cap\ldots\cap A_{i_k}\right].\end{aligned}$$

(b) We want to use induction. The base case are clearly true, i.e., when $n = 1$ both formulas are true.For the inductive step suppose the formulas are true for an integer n and we will try to prove them for $n + 1$. The first one:

$$\begin{aligned}
\mathbb{P}\left[\bigcup_{j=1}^{n+1} A_j\right] &= \mathbb{P}\left[\bigcup_{j=1}^{n} A_j \cup A_{n+1}\right] \\
&= \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] + \mathbb{P}\left[A_{n+1}\right] - \mathbb{P}\left[\bigcup_{j=1}^{n} A_j \cap A_{n+1}\right] \\
&\leq \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] + \mathbb{P}\left[A_{n+1}\right] - \mathbb{P}\left[A_n \cap A_{n+1}\right] \\
&\leq \sum_{j=1}^{n} \mathbb{P}[A_j] - \sum_{j=1}^{n-1} \mathbb{P}\left[A_j \cap A_{j+1}\right] + \mathbb{P}\left[A_{n+1}\right] - \mathbb{P}\left[A_n \cap A_{n+1}\right] \\
&= \sum_{j=1}^{n+1} \mathbb{P}[A_j] - \sum_{j=1}^{n} \mathbb{P}\left[A_j \cap A_{j+1}\right].
\end{aligned}$$

For the second formula we have:

$$\begin{aligned}
\mathbb{P}\left[\bigcup_{j=1}^{n+1} A_j\right] &= \mathbb{P}\left[\bigcup_{j=1}^{n} A_j \cup A_{n+1}\right] \\
&= \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] + \mathbb{P}\left[A_{n+1}\right] - \mathbb{P}\left[\bigcup_{j=1}^{n} (A_j \cap A_{n+1})\right] \\
&\geq \mathbb{P}\left[\bigcup_{j=1}^{n} A_j\right] + \mathbb{P}\left[A_{n+1}\right] - \sum_{j=1}^{n} P(A_j \cap A_{n+1}) \\
&\geq \sum_{j=1}^{n} \mathbb{P}[A_j] - \sum_{i,j=1,i<j}^{n} \mathbb{P}\left[A_j \cap A_i\right] + \mathbb{P}\left[A_{n+1}\right] - \sum_{j=1}^{n} P(A_j \cap A_{n+1}) \\
&= \sum_{j=1}^{n+1} \mathbb{P}[A_j] - \sum_{i,j=1,i<j}^{n+1} \mathbb{P}\left[A_j \cap A_i\right].
\end{aligned}$$

Q4. Show the Multiplication Rule for Conditional Probabilities:

Suppose that $A_1, A_2, \ldots, A_n$ are events such that $\mathbb{P}(A_1 \cap \cdots \cap A_{n-1}) > 0$. Then

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\cdots\mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1}).$$

Solution

We use the definition of conditional probabilities: if $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B),$$

therefore $\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A|B)$. Apply it to $A = A_n$ and $B = A_1 \cap A_2 \cap \cdots \cap A_{n-1}$, we get

$$\mathbb{P}(A_1 \cap \cdots \cap A_n) = \mathbb{P}(A_1 \cap \cdots \cap A_{n-1})\mathbb{P}(A_n|A_1 \cap \cdots \cap A_{n-1}).$$

Since $A_1 \cap \cdots \cap A_{n-2}$ contains $A_1 \cap \cdots \cap A_{n-1}$, the former event has positive probability, we can inductively deduce the multiplication rule as announced.