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Solution Series 10

Q1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and $(Z_n)_{n\in\mathbb{N}}$ a sequence of random variables.

(a) Prove that if $Z_n \xrightarrow{\mathbb{P}} c \in \mathbb{R}$, then for all bounded and continuous functions f

$$\mathbb{E}\left(f(Z_n)\right) \to f(c).$$

(b) Show that if $Z_n \to c \in \mathbb{R}$ in distribution, then $Z_n \xrightarrow{\mathbb{P}} c$.

Solution:

(a) Take $\epsilon > 0$, we know by continuity of f that there exists $\delta > 0$ so that for all $x \in [c-\delta, c+\delta]$, $|f(x) - f(c)| \leq \epsilon$. Then

$$\begin{aligned}
|\mathbb{E}\left(f(Z_n) - f(c)\right)| &\leq \mathbb{E}\left(|f(Z_n) - f(c)|\right) \\
&\leq \mathbb{E}\left(|f(Z_n) - f(c)|\mathbf{1}_{\{|Z_n - c| \leq \delta\}}\right) + \mathbb{E}\left(|f(Z_n) - f(c)|\mathbf{1}_{\{|Z_n - c| > \delta\}}\right) \\
&\leq \epsilon + \|f\|_\infty \mathbb{P}(|Z_n - c| > \delta) \xrightarrow[n\to\infty]{} \epsilon.
\end{aligned}$$

(b) Take $\epsilon > 0$ and define

$$f_\epsilon(x) \mapsto \min\left\{\frac{1}{\epsilon} d(x, [c-\epsilon, c+\epsilon]), 1\right\}.$$

f_ϵ is clearly a continuous function. Note that $f_\epsilon(x) = 0$ if $x \in [c-\epsilon, c+\epsilon]$ and $f(x) = 1$ if $|x - c| \geq 2\epsilon$. Then we have that:

$$\mathbb{P}(|X_n - c| \geq 2\epsilon) \leq f_\epsilon(X_n) \to f_\epsilon(c) = 0.$$

Q2. Take the following probability space $(\Omega, \mathcal{A}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \lambda|_{[0,1]})$, where $\lambda|_{[0,1]}$ is the Lebesgue measure over $[0,1]$. Let $X_n(\omega) = \mathbf{1}_{A_n}(\omega)$ a sequence of random variables with $A_n \in \mathcal{B}([0,1])$.

(a) Under which condition for $(A_n)_{n\in\mathbb{N}}$ we have that $X_n \xrightarrow{\mathbb{P}} 0$.

(b) Write the event $\{\omega : X_n(\omega) \to 0\}$ with help of the sets $(A_n)_{n\in\mathbb{N}}$.

(c) Find a sequence $(A_n)_{n\in\mathbb{N}}$ of events so that $X_n \xrightarrow{\mathbb{P}} 0$ but $\{\omega : X_n(\omega) \to 0\} = \emptyset$.

Solution:

(a) We know that for all $\epsilon \leq \frac{1}{2}$

$$\mathbb{P}(|X_n| \leq \epsilon) = \mathbb{P}(|X_n| = 0) = \mathbb{P}(A_n^c),$$

so $X_n \xrightarrow{\mathbb{P}} 0$ iff $\mathbb{P}(A_n^c) \to 1$.

(b) Given that X_n takes only values in $\{0, 1\}$ we know it converges if from a point onward it only takes the value 0, so

$$\{\omega : \lim X_n(\omega) = 0\} = \bigcup_{k \in \mathbb{N}} \bigcap_{n \geq k} A_n^c = \liminf A_n^c.$$

(c) For $n \in \mathbb{N}$ define $r_n = \lfloor \log_2(n) \rfloor$ and define $k_n = n - 2^{r_n}$. Take

$$A_n = \left[\frac{k_n}{2^{r_n}}, \frac{k_n + 1}{2^{r_n}}\right],$$

note that $\mathbb{P}(A_n) = r_n \to 0$, so $X_n \xrightarrow{\mathbb{P}} 0$. Additionally note that for each r_n there are $2^{r_n+1} - 2^{r_n} = 2^{r_n}$ different k_n associated to it and also that:

$$\mathbb{P}\left(\bigcup_{n:r_n=r} A_n\right) = 2^{r_n} \frac{1}{2^{r_n}} = 1,$$

so $\bigcup_{n:r_n=r} A_n = [0, 1]$. Then we know that for each $r \in \mathbb{N}$ and for all $x \in [0, 1]$ there exits $n \in \mathbb{N}$ so that $r_n = r$ and $x \in A_n$, so $X_n(x)$ is 1 infinitely many times. Thus, $\{\omega : X_n(\omega) \to 0\} = \emptyset$.

Q3. Let $(X_i)_{i \geq 1}$ be a sequence of random variables with

$$\begin{aligned}
&\mathbb{E}(X_i) = \mu \quad \forall i, \\
&Var(X_i) = \sigma^2 < \infty \quad \forall i, \\
&Cov(X_i, X_j) = R(|i - j|) \quad \forall i, j.
\end{aligned}$$

Define $S_n := \sum_{i=1}^{n} X_i$.

(a) Prove that if $\lim_{k\to\infty} R(k) = 0$ then $\lim_{n\to\infty} \frac{S_n}{n} = \mu$ in probability.

(b) Prove that if $\sum_{k\in\mathbb{N}} |R(k)| < \infty$ then $\lim_{n\to\infty} nVar(\frac{S_n}{n})$ exists.

Solution:

(a) Thanks to Chebyshev inequality

$$P\left[\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right] \leq \frac{1}{\varepsilon^2} Var\left(\frac{S_n}{n}\right)$$

it's enough to prove that $Var(\frac{S_n}{n}) \to 0 \, (n \to \infty)$.

Computing the variance we have:

$$\begin{aligned} Var\left(\frac{S_n}{n}\right) &= Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2}\left(\sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j}\text{Cov}(X_i, X_j)\right) \\ &= \frac{1}{n^2}\left(n\sigma^2 + 2\sum_{k=1}^{n-1}(n-k)\,R(k)\right) \\ &= \frac{1}{n}\left(\sigma^2 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)R(k)\right) \end{aligned}$$

Then it's enough to prove that:

$$\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n-1}\Big(\frac{n-k}{n}\Big)R(k) = 0\,,$$

which is obtained by a similar proof of the convergence of Cesàro means.

(b) We just have to compute

$$\begin{aligned} \lim_{n\to\infty} n\,Var\left(\frac{S_n}{n}\right) &= \lim_{n\to\infty}\left(\sigma^2 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)R(k)\right) \\ &= \sigma^2 + 2\sum_{k=1}^{\infty} R(k) - 2\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{k}{n}R(k)\,. \end{aligned}$$

Define:

$$a_n(k) := \begin{cases} \frac{k}{n}R(k) & (k<n) \\ 0 & (k\geq n) \end{cases}$$

it's clear that $a_n(k) \to 0$ $(n\to\infty)$ for all k. Then we just have to use dominated convergence to prove that this part goes to 0. Note that $|a_n(k)| \leq |R(k)|$and $|R(k)|$ is absolutely convergente. So:

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{k}{n}R(k) = \lim_{n\to\infty}\sum_{k=1}^{n-1}a_n(k) = \lim_{n\to\infty}\sum_{k=1}^{\infty}a_n(k) = \sum_{k=1}^{\infty}\lim_{n\to\infty}a_n(k) = 0$$

Then

$$\lim_{n\to\infty} n\,Var\Big(\frac{S_n}{n}\Big) = \sigma^2 + 2\sum_{k=1}^{\infty}R(k).$$

Q4. **(a)** Let μ_n and ν_n two sequence of probability measure on $\mathbb{R}$. and $\epsilon_n \in (0,1)$ with $\epsilon_n \to 0$. Prove that if $\mu_n \to \mu$ in distribution, then $(1-\epsilon_n)\mu_n + \epsilon_n\nu_n \to \mu$ in distribution.

(b) Construct with the help of a) a sequence μ_n so that $\mu_n \to \mu$ in distribution but $\lim_{n\to\infty} \int |x| d\mu_n(x) \neq \int |x| d\mu(x)$.

(c) Prove that if $\mu_n \to \mu$ in distribution and $\sup_n \int x^2 d\mu_n(x) = K < \infty$ then

$$\int |x| d\mu_n(x) \to \int |x| d\mu(x).$$

HINT: For all M prove that

$$\int \min\{|x|, M\} d\mu_n(x) \to \int \min\{|x|, M\} d\mu(x).$$

and that

$$0 \leq \int |x| d\mu_n(x) - \int \min\{|x|, M\} d\mu_n(x) \leq K/M.$$

Solution:

(a) Take $f : \mathbb{R} \to \mathbb{R}$ a continuous and bounded function

$$\begin{aligned} \left| \int f d((1-\epsilon_n)\mu_n + \epsilon_n \nu_n) - \int f d\mu \right| &\leq \left| \int f d\mu_n - \int f d\mu \right| + \epsilon_n \left| \int f d\nu_n - \int f d\mu_n \right| \\ &\leq \left| \int f d\mu_n - \int f d\mu \right| + 2\epsilon_n \|f\|_\infty \to 0. \end{aligned}$$

(b) Take $\mu_n = \delta_0$, i.e. $\mu(A) = \mathbf{1}_{\{0 \in A\}}$ and $\nu_n = \delta_n$. It's clear that $\mu_n \to \delta_0$ (it's a constant sequence), so $\left(1 - \frac{1}{n}\right) \mu_n + \frac{1}{n} \nu_n \to \delta_0$, but:

$$\int |x| d\left(\left(1 - \frac{1}{n}\right) \mu_n + \frac{1}{n} \nu_n\right)(x) = \frac{1}{n} n = 1 \neq 0 = \int |x| d\delta_0(x).$$

(c) We prove first both claims in the Hint. We know that $\min\{|\cdot|, M\}$ is a bounded continuous function. So it's clear that

$$\int \min\{|x|, M\} d\mu_n(x) \to \int \min\{|x|, M\} d\mu(x),$$

and

$$\begin{aligned}
&\int |x| d\mu_n(x) - \int \min\{|x|, M\} d\mu_n(x) \\
&= \int (|x| - M) \mathbf{1}_{|x| \geq M} d\mu_n(x) \\
&\leq \int |x| \mathbf{1}_{|x| \geq M} d\mu_n(x) \\
&\leq \sqrt{\int x^2 d\mu_n(x) \int \mathbf{1}_{|x| \geq M} d\mu_n(x)} \\
&\leq \sqrt{K} \sqrt{\int \mathbf{1}_{|x|^2 \geq M^2} d\mu_n(x)} \\
&\leq \sqrt{K} \sqrt{K/M^2} \\
&= K/M
\end{aligned}$$

thanks to Cauchy-Schwarz inequality and Chebychev inequality. The above difference is clearly non-negative.
By the monotone convergence theorem

$$\int \min\{|x|, M\} d\mu(x) \overset{M \to \infty}{\nearrow} \int |x| d\mu(x)$$

To finish, take $\epsilon > 0$, and M so that $K/M \leq \epsilon$, and that

$$\left| \int \min\{|x|, M\} d\mu(x) - \int |x| d\mu(x) \right| \leq \epsilon.$$

Take n_0 such that for all $n \geq n_0$,

$$\left| \int \min\{|x|, M\} d\mu_n(x) - \int \min\{|x|, M\} d\mu(x) \right| \leq \epsilon.$$

Finally,

$$\begin{aligned}
&\left| \int |x| d\mu_n(x) - \int |x| d\mu(x) \right| \\
\leq &\left| \int |x| d\mu_n(x) - \int \min\{|x|, M\} d\mu_n(x) \right| + \left| \int \min\{|x|, M\} d\mu_n(x) - \int \min\{|x|, M\} d\mu(x) \right| \\
&+ \left| \int \min\{|x|, M\} d\mu(x) - \int |x| d\mu(x) \right| \\
\leq &K/M + \epsilon + \epsilon = 3\epsilon.
\end{aligned}$$

Since ϵ is arbitrary we get the convergence.