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# Solution Series 11

**Q1.** Suppose that $X_1, \cdots, X_n$ form a random sample from a Poisson distribution for which the mean is unknown. Determine the maximum-likelihood estimator of the standard deviation of the distribution.

**Solution:**

The Poisson distribution with mean $\lambda$ has variance $\lambda$, thus the standard deviation is $\sqrt{\lambda}$. Given the parameter $\lambda$, the p.f. of the Poisson distribution is

$$\mathbb{P}(X = k|\lambda) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Hence the M.L.E. is the value $\lambda$ which maximizes

$$f(\lambda) = \prod_{i=1}^{n} e^{-\lambda}\frac{\lambda^{X_i}}{X_i!} = \frac{(e^{-\lambda}\lambda^{\bar{X}})^n}{X_1! \dots X_n!}.$$

Where $\bar{X} = (X_1 + \cdots + X_n)/n$. We need to find the $\lambda$ which maximizes

$$g(\lambda) = e^{-\lambda}\lambda^{\bar{X}} = \exp(-\lambda + \bar{X}\ln(\lambda)).$$

$$g'(\lambda) = (-1 + \bar{X}/\lambda)g(\lambda).$$

The maximum of $g$ is reached when $\lambda = \bar{X}$. Thus the M.L.E. of the standard deviation is $\sqrt{\bar{X}}$.

**Q2.** Suppose that $X_1, \cdots, X_n$ form a random sample from a distribution for which the p.d.f. $f(x|\theta)$ is as follows:

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Also, suppose that the value of $\theta$ is unknown ($\theta > 0$). Find the M.L.E. of $\theta$.

**Solution:**

Let

$$L(\theta) := \prod_{i=1}^{n} \theta X_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} X_i\right)^{\theta-1}.$$

The derivative of $\ln(L)(\theta)$ is:

$$\left[n\ln(\theta)+(\theta-1)\ln\left(\prod_{i=1}^{n}X_i\right)\right]'=\frac{n}{\theta}+\ln\left(\prod_{i=1}^{n}X_i\right).$$

Thus

$$\theta_0=-\frac{n}{\ln\left(\prod_{i=1}^{n}X_i\right)}=-\frac{1}{\overline{\ln(X)}},$$

where $\overline{\ln(X)}=\frac{1}{n}\sum_{i=1}^{n}\ln(X_i)$, is a critical point of $L$. For $\theta<\theta_0$, $\ln(L)$ is increasing and for $\theta>\theta_0$, $\ln(L)$ is decreasing. Thus $\theta_0$ is the global maximum, and is the M.L.E. of $\theta$.

**Q3.** In a lake we want to estimate the amount of a certain type of fish. For this we mark 5 fishes and we let them mix with the others, when they are well mixed we fish 11, and we realize that there are 3 marked and 8 non-marked. What is the maximum-likelihood estimator for the amount of fishes?.

**Solution:**

Define $X$ the amount of marked fishes we fished. If there are $N$ fishes in the lake, the probability of $X=3$ is given by

$$\begin{aligned}\mathbb{P}_N(X=3)&=\frac{\binom{5}{3}\binom{N-5}{8}}{\binom{N}{11}}\mathbf{1}_{\{N\geq 13\}}\\&=\frac{5!(N-5)!11!(N-11)!}{3!2!8!(N-13)!N!}\mathbf{1}_{\{N\geq 13\}}:=g(N).\end{aligned}$$

We have to find $N_{\max}\in\mathbb{N}$ so that $g(N_{\max})=\sup_{N\in\mathbb{N}}g(N)$. We have that for $N\geq 13$

$$\begin{aligned}\frac{g(N)}{g(N+1)}-1&=\frac{(N-12)(N+1)}{(N-4)(N-10)}-1\\&=\frac{3(N-17,333\ldots)}{(N-4)(N-10)},\end{aligned}$$

thus,

$$\frac{g(N)}{g(N+1)}\begin{cases}\leq 1 & \text{if } N\leq 17,\\ \geq 1 & \text{if } N\geq 18.\end{cases}$$

Then $N_{\max}=18$.

**Q4.** A gas station estimates that it takes at least $\alpha$ minutes for a change of oil. The actual time varies from costumer to costumer. However, one can assume that this time will be well represented by an exponential random variable. The random variable $X$, therefore, possess the following density funciont

$$f(t)=e^{\alpha-t}\mathbf{1}_{\{t\geq\alpha\}},$$

i.e. $X = \alpha + Z$ where $Z \sim Exp(1)$. The following values were recorded from 10 clients randomly selected (the time is in minutes):

$$4.2, 3.1, 3.6, 4.5, 5.1, 7.6, 4.4, 3.5, 3.8, 4.3.$$

Estimate the parameter $\alpha$ using the estimator of maximum likelihood.

**Solution:**

We have that the likelihood function is given by:

$$\begin{aligned} L(X_1, ..., X_n, \alpha) &= \prod_{i=1}^{n} \exp(\alpha - X_i)\mathbf{1}_{\{X_i \geq \alpha\}}, \\ &= \exp(n\alpha - \sum_{i=1}^{n} X_i)\mathbf{1}_{\{\bigcap_{i=1}^{n} X_i \geq \alpha\}}, \end{aligned}$$

we note that $f(\alpha) := \exp(n\alpha - \sum_{i=1}^{n} X_i) > 0$ is increasing, so its maximum is attained at the maximum point where $\mathbf{1}_{\{\bigcap_{i=1}^{n}\{X_i \geq \alpha\}\}} \neq 0$. Then the point that maximizes the likelihood is in $\alpha = \min_{i=1,..,n}\{X_i\}$.

**Q5.** Suppose that $X_1, \cdots, X_n$ form a random sample from a normal distribution for which both the mean and the variance are unknown. Find the M.L.E. of the 0.95 quantile of the distribution, that is, of the point $\theta$ such that $\mathbb{P}(X < \theta) = 0.95$.

The 0.95 quantile of a standard normal distribution is $1.645 =: \theta_0$.

**Solution:**

Let $\mu$ and $\sigma^2$ denote the mean and the variance of $X_i$, the density of $X_i$ is

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The 0.95 quantile of $X_i$ is the value $\theta = \theta(\mu, \sigma)$ such that

$$\mathbb{P}(X < \theta) = 0.95 = \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{\theta-\mu}{\sigma}\right),$$

thus $\theta_0 = \frac{\theta-\mu}{\sigma}$ which implies that $\theta = \sigma\theta_0 + \mu$.

The ln of the product of the density of $X_i$s is

$$L(\mu, \sigma^2) := -\frac{n}{2}\ln(2\pi\sigma^2) - \sum \frac{(X_i-\mu)^2}{2\sigma^2}.$$

We look for $\mu$ and $\sigma^2$ which maximizes $L(\mu, \sigma^2)$.

$$\frac{\partial L}{\partial \mu} = \sum \frac{X_i - \mu}{\sigma^2} = 0$$

$$\frac{\partial L}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \left(\sum \frac{(X_i-\mu)^2}{2}\right)\frac{1}{(\sigma^2)^2} = 0.$$

Which give, for any $\sigma^2$, $\mu \mapsto L(\mu, \sigma^2)$ is maximized when

$$\mu = \overline{X} = \frac{1}{n}\sum X_i,$$

which is the sample average of $X_i$. And for $\mu = \overline{X}$, $\sigma^2 \mapsto L(\mu, \sigma^2)$ is maximized when

$$\sigma^2 = \sum \frac{(X_i - \overline{X})^2}{n} = \overline{(X_i - \overline{X})^2},$$

the sample variance.

We substitute the values of $\mu$ and $\sigma^2$ into the expression of $\theta$ and gives the M.L.E. of 0.95-quantile of $X$:

$$\theta = \theta_0\sqrt{\overline{(X_i - \overline{X})^2}} + \overline{X}.$$