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Solution Series 12

Q1. If a random variable X has the χ^2 distribution with m degrees of freedom, then the distribution of $X^{1/2}$ is called a *chi (χ) distribution with m degrees of freedom.* Determine the mean of this distribution.

Solution:

Let X follows Chi-squared distribution $\chi^2(m)$ with m degrees of freedom which is also a Gamma distribution with parameter $(m/2, 1/2)$. Then $\sqrt{X}$ follows $\chi(m)$, has the expected value:

$$\begin{aligned}\mathbb{E}(\sqrt{X}) &= \frac{(1/2)^{m/2}}{\Gamma(m/2)}\int_0^\infty \sqrt{x}x^{m/2-1}e^{-x/2}dx \\ &= \frac{(1/2)^{m/2}}{\Gamma(m/2)}\int_0^\infty y^{(m+1)/2-1}e^{-y}(1/2)^{-(m+1)/2}dy \\ &= \sqrt{2}\frac{\Gamma((m+1)/2)}{\Gamma(m/2)}.\end{aligned}$$

Q2. Moore and McCabe (1999) describe an experiment conducted in Australia to study the relationship between taste and the chemical composition of cheese. One chemical whose concentration can affect taste is lactic acid. Cheese manufacturers who want to establish a loyal customer purchases the cheese. The variation in concentrations of chemicals like lactic acid can lead to variation in the taste of cheese. Suppose that we model the concentration of lactic acid in several chunks of cheese as independent normal random variables with mean μ and variance $\sigma^2 = 0.09$.

Suppose that we will sample 20 chunks of cheese. Let

$$T = \sum_{i=1}^{20}(X_i - \mu)^2/20,$$

where X_i is the concentration of lactic acid in ith chunk. What number c satisfies $\mathbb{P}(T \leq c) = 0.9$?

Solution:

$$T = \frac{\sigma^2}{20}\sum_{i=1}^{20}\left(\frac{X_i - \mu}{\sigma}\right)^2.$$

To find the 0.9-quantile of T, remark that

$$\mathbb{P}(T \leq c) = \mathbb{P}\left(\frac{20}{\sigma^2}T \leq \frac{20}{\sigma^2}c\right).$$

Where $\frac{20}{\sigma^2}T \sim \chi^2(20)$. We find in the table of inverse c.d.f. of Chi-squared distribution that the 0.9-quantile for $\chi^2(20)$ is 28.41. Hence

$$\frac{20}{\sigma^2}c = 28.41 \Rightarrow c = 28.41 * 0.09/20 = 0.128.$$

Q3. When the motion of a microscopic particle in a liquid or a gas is observed, it is seen that the motion is irregular because the particle collides frequently with other particles. The probability model for this motion, which is called *Brownian motion*, is as follows: A coordinate system is chosen in the liquid or gas. Suppose that the particle is at the origin of this coordinate system at time $t = 0$, and let (X, Y, Z) denote the coordinates of the particle at any time $t > 0$. The random variables X, Y, and Z are i.i.d., and each of them has the normal distribution with mean 0 and variance $\sigma^2 t$. Find the probability that at time $t = 2$ the particle will lie within a sphere whose center is at the origin and whose radius is 4σ.

Solution:

At time $t = 2$, X, Y, Z are i.i.d. normal distribution with mean 0 and variance $2\sigma^2$. Let $\|x, y, z\|$ denote the euclidean distance from (x, y, z) to 0,

$$\mathbb{P}(\|X, Y, Z\| \leq 4\sigma) = \mathbb{P}((X^2 + Y^2 + Z^2)/2\sigma^2 \leq 8).$$

Since $X^2 + Y^2 + Z^2 = 2\sigma^2[(X/\sqrt{2}\sigma)^2 + (Y/\sqrt{2}\sigma)^2 + (Z/\sqrt{2}\sigma)^2]$, we have

$$\frac{X^2 + Y^2 + Z^2}{2\sigma^2} \sim \chi^2(3).$$

The distribution of $\chi^2(3)$ is a Gamma distribution with parameter $(3/2, 1/2)$, with p.d.f.

$$f(x) = \frac{(1/2)^{3/2}}{\Gamma(3/2)}x^{1/2}e^{-x/2}.$$

We find that the probability of the particle to be in the ball of radius 4σ of the origine is the probability of a $\chi^2(3)$ being less than 8:

$$\mathbb{P}((X^2 + Y^2 + Z^2)/2\sigma^2 \leq 8) = \frac{(1/2)^{3/2}}{\Gamma(3/2)}\int_0^8 x^{1/2}e^{-x/2}dx.$$

Q4. Take $x \in [0, 1]$. We say that x is normal if for x in its binary form,

$$x = \sum_{n\in\mathbb{N}} x_n 2^{-n} \qquad x_n \in \{0, 1\},$$

we have that $\lim_{n\to\infty} \frac{|\{1\leq k\leq n : x_k = 1\}|}{n} = \frac{1}{2}$.

(a) Prove that if we have a sequence $(U_n)_{n\in\mathbb{N}}$ i.i.d. Bernoulli with parameter $\frac{1}{2}$, then $U = \sum_{n\in\mathbb{N}} U_n 2^{-n}$ is an uniform random variable in $[0,1]$

(b) Prove that if $U \sim U(0,1)$, $\mathbb{P}(U \text{ is normal}) = 1$.

Solution:

(a) First we have to prove that U is measurable. For this we just have to realize that $U^{(m)} := \sum_{n=1}^{m} U_n 2^{-n}$ is measurable because it's the finite sum of measurable function and we have that

$$U_n \to U.$$

Second we have to understand the measure that U produces in $\mathbb{R}$. For this it's enough to show that the measure induced by U coincide with the uniform measure in the intervals of the form

$$\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right],$$

for $k \in \mathbb{N} \in [0, 2^n - 1]$. This is because they generate the Borel σ-algebra.
Note that

$$\mathbb{P}((\exists n \in \mathbb{N})(\forall m \geq n) X_m = 1) = 0,$$

thanks to Borel-Cantelli Lemma. So we can work in

$$\tilde{\Omega} := \Omega \backslash \{\omega \in \Omega : (\exists n \in \mathbb{N})(\forall m \geq n) X_m = 1\},$$

i.e. our probability space is $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$ where $\tilde{\mathcal{A}} = \mathcal{A} \mid_{\tilde{\Omega}}$ and $\tilde{\mathbb{P}} := \mathbb{P} \mid_{\tilde{\Omega}}$. Now we have that if $k = \sum_{i=0}^{n-1} k_i 2^i \in \{0, 1, .., 2^n - 1\}$:

$$\begin{aligned}
\mathbb{P}\left(U \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right) &= \mathbb{P}\left(\bigcap_{i=0}^{2^n-1} \{U_{i+1} = k_{n-i}\}\right) \\
&= \prod_{i=0}^{2^n-1} \mathbb{P}(U_{i+1} = k_{n-i}) \\
&= 2^{-n}.
\end{aligned}$$

That is the probability of a uniform random variable to be in $\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$.

(b) Take $(U_n)_{n\in\mathbb{N}}$ Bernoulli $\frac{1}{2}$ i.i.d. Thanks to part (a) we have that $U := \sum_{n=1}^{n} U_n 2^{-n}$ is uniform distributed and it's normal iff

$$\frac{\sum_{k=1}^{n} \mathbf{1}_{\{U_k=1\}}}{n} \to \frac{1}{2}.$$

Then:

$$\mathbb{P}(U \text{ is normal}) = \mathbb{P}\left(\frac{\sum_{k=1}^{n} \mathbf{1}_{\{U_k=1\}}}{n} = \frac{1}{2}\right) = 1.$$

Where in the last equality we have used the law of large numbers.