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Solution Series 13

Q1. We toss 100 times a coin and we get 60 head. We want to do a test to know whether the coin is fair.

(a) Test the hypothesis with a 0.01 level of significance. Should this test be one or two-tailed?.

(b) What is the biggest amount of head should we have in 100 tossings so we cannot discard H_0 := "The coin biased towards tail".

(c) Calculate all p_0 so that the null hypothesis

$$H_0(p_0) := \text{"Probability of head is } p_0 \text{ "},$$

would not be rejected in a test with 0.05 level of significance.
Hint: It will be useful to use the central limit theorem in all of this question.

Solution:

Let $(X_i)_{i=1}^n$ an i.i.d sequence of Bernoulli(p). Let $X = \sum_{i=1}^{100} X_i$, then $X \sim Bin(100, p)$.

(a) We want to know whether the coin is fair or not, so our hypothesis are

$$H_0 : \text{The coin is fair,}$$
$$H_1 : \text{The coin is not fair.}$$

That is to say

$$H_0 : p = p_0 = \frac{1}{2},$$
$$H_1 : p \neq p_0.$$

Then we should use a two-tailed test.
Under H_0 we have that $\mathbb{E}_0(X) = 50$ and $Var_0(X) = np_0(1-p_0) = 25$. By central limit theorem, the distribution of $\frac{X-50}{5}$ can be approximated by the standard normal distribution. Let ϕ be the c.d.f. of the standard normal distribution. Now, take c_1 and c_2 such that

$$\begin{aligned} 0.01 &\geq \mathbb{P}_0(X \notin (c_1, c_2)) \\ &= 1 - \mathbb{P}_0\left(\frac{c_1 - 50}{5} \leq \frac{X-50}{5} \leq \frac{c_2 - 50}{5}\right) \\ &\sim 1 - \phi\left(\frac{c_2 - 50}{5}\right) + \phi\left(\frac{c_1 - 50}{5}\right). \end{aligned}$$

We would like to make the rejection zone as biggest as we can, given that ϕ is symmetric, we just have to make

$$\begin{aligned}\frac{c_1-50}{5} &= -\frac{c_2-50}{5},\\ \Rightarrow c_1 &= 100-c_2.\end{aligned}$$

Finally

$$\begin{aligned}0.01 &\geq 1-\phi\left(\frac{c_2-50}{5}\right)+1-\phi\left(\frac{c_2-50}{5}\right),\\ \Rightarrow \phi\left(\frac{c_2-50}{5}\right) &\geq 0.995\\ \Rightarrow \frac{c_2-50}{5} &\geq 2.5758\\ \Rightarrow c_2 &\geq 62.9,\end{aligned}$$

then

$$K_{1\%}=[0;37]\cup[63;100].$$

Given that $60\notin K_{1\%}$ we cannot reject H_0 with 0.01 level of significance.

(b) We have to take our hypothesis

$$\begin{aligned}H_0 &: p=p_0=\frac{1}{2}\\ H_1 &: p>p_0,\end{aligned}$$

now we have to find c such that

$$0.01 \geq \mathbb{P}_0(X\geq c)\sim 1-\phi\left(\frac{c-50}{5}\right).$$

Then we have to choose $c\geq 61.6$, from what we have that $K_{1\%}=[62;100]$. So we reject H_0 if we have 62 or more heads.

(c) Take $p_0\in(0,1)$ and

$$\begin{aligned}H_0 &: p=p_0,\\ H_1 &: p\neq p_0.\end{aligned}$$

Note that $\mathbb{E}_0(X)=100p_0$ and $Var_0(X)=100p_0(1-p_0)$. Then, take c_1 and c_2 such that

$$\begin{aligned}0.01 &\geq \mathbb{P}_0(X\notin(c_1,c_2))\\ &= 1-\mathbb{P}_0\left(\frac{c_1-100p_0}{10\sqrt{p_0(1-p_0)}}\leq\frac{X-100p_0}{10\sqrt{p_0(1-p_0)}}\leq\frac{c_2-100p_0}{10\sqrt{p_0(1-p_0)}}\right)\\ &\sim 1-\phi\left(\frac{c_2-100p_0}{10\sqrt{p_0(1-p_0)}}\right)+\phi\left(\frac{c_1-100p_0}{10\sqrt{p_0(1-p_0)}}\right).\end{aligned}$$

Doing the same that in question a we have that $c_1(p_0) = 100p_0 - 19.6\sqrt{p_0(1-p_0)}$ and $c_2(p_0) = 100p_0 + 19.6\sqrt{p_0(1-p_0)}$. Then we are interested in the set

$$
\begin{aligned}
&\left\{p : 100p - 19.6\sqrt{p(1-p)} \leq 60 \leq 100p + 19.6\sqrt{p(1-p)}\right\} \\
=&\left\{p : (60 - 100p)^2 \leq 19.6^2 p(1-p)\right\} \\
=&[0.502; 0.691].
\end{aligned}
$$

Let δ be a test procedure and S_1 the critical region of δ, Θ the parameter space. The function $\pi(\theta|\delta)$, called the *power function* of the test δ, is determined by the relation

$$\pi(\theta|\delta) = \mathbb{P}(X \in S_1|\theta), \quad \text{for } \theta \in \Theta.$$

Suppose that $X_1, \cdots, X_n$ form a random sample from the normal distribution with unknown mean μ and known variance 1, and it is desired to test the following hypotheses:

$$
\begin{aligned}
H_0 &: \quad 0.1 \leq \mu \leq 0.2 \\
H_1 &: \quad \mu < 0.1 \text{ or } \mu > 0.2.
\end{aligned}
$$

Consider a test procedure δ such that the hypothesis H_0 is rejected if either $\overline{X_n} \leq c_1$ or $\overline{X_n} \geq c_2$, and let $\pi(\mu|\delta)$ denote the power function of δ. Suppose that the sample size is $n = 25$. Determine the values of the constants c_1 and c_2 such that

$$\pi(0.1|\delta) = \pi(0.2|\delta) = 0.07.$$

Solution: Given $n = 25$, the mean of X_i is μ and variance is 1,

$$\overline{X_n} \sim \mathcal{N}(\mu, \sigma^2/n) = \mathcal{N}(\mu, 1/25).$$

Thus $5(\overline{X_n} - \mu) \sim \mathcal{N}(0, 1)$. Let Y be a standard normal distributed random variable and Φ its c.d.f.:

$$
\begin{aligned}
\pi(0.1|\delta) &= \mathbb{P}(\overline{X_n} \leq c_1|\mu = 0.1) + \mathbb{P}(\overline{X_n} \geq c_2|\mu = 0.1) \\
&= \mathbb{P}(5\overline{X_n} - 0.5 \leq 5c_1 - 0.5|\mu = 0.1) + \mathbb{P}(5\overline{X_n} - 0.5 \geq 5c_2 - 0.5|\mu = 0.1) \\
&= \mathbb{P}(Y \leq 5c_1 - 0.5) + \mathbb{P}(Y \geq 5c_2 - 0.5) \\
&= \Phi(5c_1 - 0.5) + (1 - \Phi(5c_2 - 0.5)).
\end{aligned}
$$

Similarly,

$$\pi(0.2|\delta) = \Phi(5c_1 - 1) + (1 - \Phi(5c_2 - 1)).$$

We have to find c_1 and c_2 such that

$$\Phi(5c_2 - 0.5) - \Phi(5c_1 - 0.5) = 0.93 = \Phi(y + 0.5) - \Phi(x + 0.5) \tag{1}$$

and

$$\Phi(5c_2 - 1) - \Phi(5c_1 - 1) = 0.93 = \Phi(y) - \Phi(x) \tag{2}$$

where we set $x = 5c_1 - 1$ and $y = 5c_2 - 1$.

It is not hard to see that $x + 0.5 = -y$. In fact, taking the difference of (1) and (2), one gets:

$$\Phi(y + 0.5) - \Phi(y) = \Phi(x + 0.5) - \Phi(x).$$

By studying the monotonicity of the function $t \mapsto \Phi(t + 0.5) - \Phi(t)$, any value can have at most two preimages, and the fact that $0.93 = \Phi(y) - \Phi(x)$ implies $x \neq y$. Hence $x + 0.5 = -y$. Substitute it in (2):

$$0.93 = 1 - \Phi(-y) - \Phi(x) = 1 - \Phi(x + 0.5) - \Phi(x)$$

$$0.07 = \Phi(x + 0.5) + \Phi(x).$$

It has a unique solution $x = x_0$ (by monotonicity of $x \mapsto \Phi(x + 0.5) + \Phi(x)$). And $y = -x_0 - 0.5$.

Hence $c_1 = x_0/5 + 1/5$ and $c_2 = -x_0/5 + 1/10$.