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Solution Series 3

Q1. In a clinical trial with two treatment groups, the probability of success in one treatment group is 0.5, and the probability of success in the other is 0.6. Suppose that there are five patients in each group. Assume that the outcomes of all patients are independent. Calculate the probability that the first group will have at least as many successes as the second group.

Solution:

The number of successes in the first group follows the Binomial law $\mathcal{B}(5, 0.5)$, and in the second group follows $\mathcal{B}(5, 0.6)$. We check in the table of Binomial Probabilities to find:

	k=0	k=1	k=2	k=3	k=4	k=5
p=0.5	0.0312	0.1562	0.3125	0.3125	0.1562	0.0312
p=0.6	0.0102	0.0768	0.2304	0.3456	0.2592	0.0778

The probability that the first group has at least as many successes as the second group is given by

$$0.03120 * 0.0102 + 0.1562 * (0.0102 + 0.0768) + ... + 0.0312 * 1.$$

Q2. Recall that a *Poisson process* with rate λ per unit time is a process that satisfies two properties:

(a) The number of arrivals in every fixed interval of time of length t has the Poisson distribution with mean λt.

(b) The numbers of arrivals in every collection of disjoint time intervals are independent.

Suppose that the arrival time of clients to store A is a Poisson process with rate 1 per hour.

(a) What is the probability of "the first client comes later than t hours"?

(b) What is the probability of "more than two clients come during the first hour"?

(c) Fix a time $t > 0$, what is the probability of "a client comes at some exact time t"? Does it mean that nobody comes at any time? Is that contradictory?

(d) Suppose that the arrival time of clients to another store B is a Poisson process with rate μ per hour which is independent to the store A, now assumed to have rate λ. What are arrival times of clients to both store (if we forget about in which store they are)?

Solution:

(a) The event "the first client comes later than t hours" is the same as "nobody comes during the first t hours". The number of visits during the first t hours has the Poisson distribution with mean t hence nobody comes with probability e^{-t}. So the arrival time of the first client follows the exponential law of parameter 1.

(b) Similarly, the number of visits during the first hour has the law Poisson(1). The probability of having more than two clients is given by

$$1 - e^{-1}\left(\frac{1^0}{0!} + \frac{1^1}{1!} + \frac{1^2}{2!}\right) = 1 - 5/(2e) \approx 0.0803.$$

(c) This event is included in "at least one client comes in the time interval $[t-\varepsilon/2, t+\varepsilon/2]$", which has probability

$$1 - e^{-\varepsilon} \sim \varepsilon$$

as $\varepsilon \to 0$. Thus the probability that a client comes at the exact time t is 0.

Although "a client comes during the first hour" is the union over all $t \leq 1$ of "a client comes at time t", the former event has a positive probability as we have seen in the previous question. It is not contradictory because its probability is not the sum of the probability of uncountable family of events.

(d) We show that the sum X of two independent Poisson law of parameter a and b is the Poisson$(a+b)$. Indeed, by independence:

$$\begin{aligned}
\mathbb{P}(X = k) &= \sum_{i,j\geq 0, i+j=k} e^{-a-b}\frac{a^i}{i!}\frac{b^j}{j!} \\
&= \sum_{i=0}^{k} e^{-a-b}\frac{a^i}{i!}\frac{b^{k-i}}{(k-i)!} \\
&= e^{-a-b}\frac{b^k}{k!}\sum_{i=0}^{k}\left(\frac{a}{b}\right)^i \frac{k!}{i!(k-i)!} \\
&= e^{-a-b}\frac{b^k}{k!}\left(1 + \frac{a}{b}\right)^k \\
&= e^{-a-b}\frac{(a+b)^k}{k!}.
\end{aligned}$$

Thus the total number of clients coming to both stores during time interval of length t has the poisson law $(\lambda + \mu)t$, which means the arrivals of clients is a Poisson process with rate $\lambda + \mu$.

Q3. Borel Cantelli

(a) Construct a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and a series of measurable sets $(A_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$ and $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}}\bigcup_{k\geq n} A_k\right) = 0$.

(b) Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. Take $(U_n)_{n\in\mathbb{N}}$ a series of uniform independent random variables on $(0,1)$, i.e., for $0 \leq x \leq 1$, $\mathbb{P}(U_n \in [0,x]) = x$.

i. Show that:

$$\mathbb{P}\left((\exists\alpha > 1)\liminf n^{\alpha}U_n \in \mathbb{R}\right) = 0.$$

Hint: It may be useful to define, for $\alpha > 1$ $A_n^{\alpha} := \{U_n < n^{-\alpha}\}$. Do not forget that the countable union of sets of probability 0 has probability 0.

ii. Prove that:

$$\mathbb{P}\left(\liminf nU_n \in \mathbb{R}\right) > 0.$$

Solution

(a) Take $([0,1], \mathcal{B}(0,1), \lambda)$ as a probability space and U the identity function. U is distributed as an uniform random variable on $(0,1)$. Define

$$A_n := \{x \in (0,1) : U(x) \in [0, \frac{1}{n}]\}.$$

Then we have that $\mathbb{P}(A_n) = \frac{1}{n}$, so $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$. Additionally $x \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$ iff $x = 0$, so $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right) = 0$.

(b) i. We will use Borel-Cantelli. Define $A_n^{\alpha} := \{U_n < n^{-\alpha}\}$, then:

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n^{\alpha}) = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} < \infty,$$

so $\mathbb{P}(\bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^{\alpha}) = 0$. Thus

$$\mathbb{P}\left(\bigcup_{\substack{\alpha>1\\ \alpha\in\mathbf{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^{\alpha}\right) = 0.$$

Let $\omega \in \Omega$ so that there exists $\alpha(\omega)$ for which $\liminf n^{\alpha(\omega)} U_n(\omega) < \infty$.Then take $1 < \tilde{\alpha}(\omega) < \alpha(\omega)$ with $\tilde{\alpha}(\omega) \in \mathbf{Q}$. We have that $\liminf n^{\tilde{\alpha}(\omega)} U_n(\omega) = 0$. Then for all $n \in \mathbb{N}$ there exists $m(\omega) > n$ so that $m^{\tilde{\alpha}} U_m(\omega) < 1$. Thus, $\omega \in \bigcup_{\substack{\alpha>1\\ \alpha\in\mathbf{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^{\alpha}$. Finally we have that

$$\{(\exists\alpha > 1)\liminf n^{\alpha}U_n \in \mathbb{R}\} \subseteq \bigcup_{\substack{\alpha>1\\ \alpha\in\mathbf{Q}}} \bigcap_{n\in\mathbb{N}} \bigcup_{j\geq n} A_j^{\alpha}$$

$$\Rightarrow \mathbb{P}\left((\exists\alpha > 1)\liminf n^{\alpha}U_n \in \mathbb{R}\right) = 0.$$

ii. We will use Borel-Cantelli. Define $A_n = \{U_n \leq n^{-1}\}$, it's clear that $(A_n)_{n\in\mathbb{N}}$ are independent. We have $\mathbb{P}(A_n) = \frac{1}{n}$, then $\sum_{n\in\mathbb{N}} \mathbb{P}(A_n) = \infty$. By Borel-Cantelli

$$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k\right) = 1 > 0.$$

Additionally, if $\omega \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$, for all $n \in \mathbb{N}$ there exists $k_n(\omega) > n$ so that $k_n(\omega) U_{k_n(\omega)} \leq 1$. Thus, $0 \leq \liminf nU_n \leq 1$. To conclude:

$$\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k \subseteq \{\liminf nU_n \in \mathbb{R}\}$$
$$\Rightarrow \mathbb{P}(\liminf nU_n \in \mathbb{R}) = 1 > 0.$$

Q4. THE BERTRAND'S PARADOX Consider an equilateral triangle inscribed in a circle of radius 1. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?. For solving this try the following probability models:

(a) The "random endpoints" method: Choose two uniform random points on the circumference of the circle and draw the chord joining them, i.e., let $U, V \sim U(0,1)$, define $X = e^{U2\pi i}, Y = e^{V2\pi i}$ and take the chord connecting X and Y.

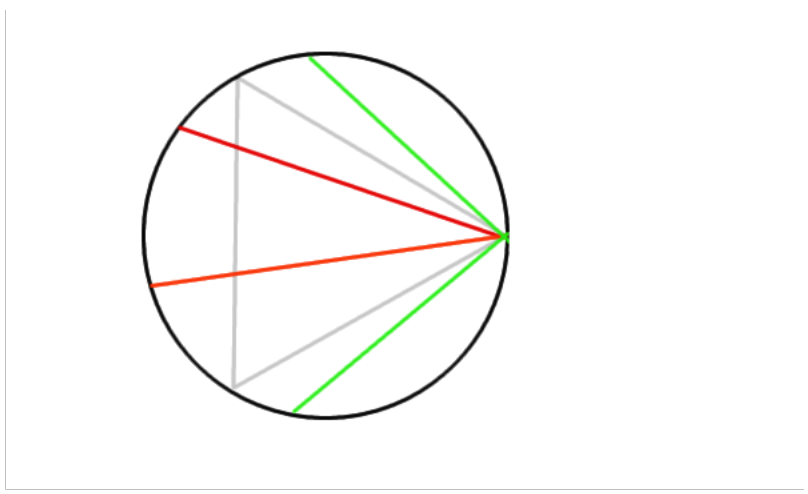
(b) The "random radius" method: Choose a radius of the circle, choose a uniform point on the radius and construct the chord through this point and perpendicular to the radius, i.e., choose a radius and choose $r \sim U(0,1)$, take the point on the radius that is at distance r from the center and the chord will be the only chord perpendicular to this point.

(c) The "random midpoint" method: Choose a point uniformly anywhere within the circle and construct a chord with the chosen point as its midpoint, i.e., take $(x,y) \sim U(B(0,1))$ and take the chord whose midpoint is (x,y).

(d) Is this a contradiction?

Solution

(a) Remember that $x \mod 1 = x - \lfloor x \rfloor$. Note that the chord is longer than a side of the triangle if $(V-U) \mod 1 \in \left(\frac{1}{3}, \frac{2}{3}\right)$.



$$\mathbb{P}\left[(V-U) \mod 1 \in \left(\frac{1}{3},\frac{2}{3}\right)\right] = \iint\limits_{(0,1)^2} \mathbf{1}_{\left\{(y-x) \mod 1 \in \left(\frac{1}{3},\frac{2}{3}\right)\right\}} dydx$$
$$= \int_0^1 \left(\int_0^1 \mathbf{1}_{\left\{y \in \left(\frac{1}{3}+x,\frac{2}{3}+x\right) \mod 1\right\}} dy\right) dx$$
$$= \int_0^1 \frac{1}{3} dy = \frac{1}{3}.$$

(b) Let r be the point chosen in the radius, it's a uniform random variable over $[0,1]$. The length of the chord will be given by $2\sqrt{(1-r^2)}$. Then

$$\mathbb{P}\left(2\sqrt{(1-r^2)} \geq \sqrt{3}\right) = \mathbb{P}\left(1-r^2 \geq \frac{3}{4}\right) = \mathbb{P}\left(r^2 \leq \frac{1}{4}\right) = \frac{1}{2}.$$

(c) Let (x,y) be the point chose in the circle, it is a uniform random variable over $B(0,1)$. The length of the chord will be given by $2\sqrt{1-(x^2+y^2)}$. Thus,

$$\mathbb{P}\left(2\sqrt{(1-(x^2+y^2))} \geq \sqrt{3}\right) = \mathbb{P}\left(1-(x^2+y^2) \geq \frac{3}{4}\right)$$
$$= \mathbb{P}\left((x^2+y^2) \leq \frac{1}{4}\right)$$
$$= \iint\limits_{B(0,\frac{1}{2})} \frac{1}{\pi} dxdy = \frac{1}{4}.$$

(d) This is not a contradiction. What this show us that there is not a formal way to "pick a chord uniformly at random" in this circle, so we have to define the probability measure we are interested in before asking the question about the probability of an event.