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# Solution Series 5

**Q1.** **(a)** Take $p \in [0,1]$ and $n \in \mathbb{N}\backslash\{0\}$. We say that $X \sim Bin(n,p)$ if the distribution of $X$ is

$$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k \in \{0,1,...,n\}.$$

Show that this is indeed a probability distribution using 2 different methods:

i. Calculating $\sum_k \mathbb{P}(X=k)$.

ii. Representing this probability in terms of the box model with replacement.

Calculate the expected value of $X$ using 2 different methods (the one listed above).

**(b)** Take $K, n \in \mathbb{N}$ and $N \in \mathbb{N}\backslash\{0\}$ with $K, n \leq N$. We say that a random variable $X \sim Hyp(N,K,n)$ if its distribution is given by

$$\mathbb{P}(X=k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} \qquad k \in \{\max\{0, n+K-N\}, .., \min\{n,K\}\}$$

Show that this is indeed a probability distribution using 2 different methods:

i. Calculating $\sum_k \mathbb{P}(X=k)$.
**Hint:** Calculate $(1+x)^n$ in two different ways and identify the terms.

ii. Representing this probability in the box model without replacement.

Calculate the expectation using both methods.

**Solution**

**(a)** Expectation:

i.

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=1}^{n} \frac{n(n-1)!}{(n-k)!(k-1)!} p^{(k-1)+1} (1-p)^{n-k} \\ &= np \sum_{j=0}^{n-1} \binom{n-1}{j} p^j (1-p)^{(n-1)-j}\end{aligned}$$

where we make the change of variables $j = k-1$. The sum we had is exactly the sum we calculated in the first part for a $Bin(n-1,p)$, so it is 1. Thus:

$$\mathbb{E}[X] = np.$$

ii. We know that the amount of red balls that are taken out at time $n$ in a experiment with replacement have the distribution of $X$. So

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}\left[\sum_{j=1}^{n} \mathbf{1}_{\{\text{In the } j\text{-th draw we get a red ball}\}}\right] \\ &= \sum_{j=1}^{n} \mathbb{P}\left(\{\text{In } j\text{-th draw we get a red ball}\}\right).\end{aligned}$$

The probability that in the $j$-th draw we get a red ball is $p$, so:

$$\mathbb{E}\left[X\right] = np.$$

**(b)** Expectation:

i.

$$\begin{aligned}\mathbb{E}\left[X\right] &= \sum_{k=\max\{1,N-K-k\}}^{\min\{n,K\}} k\frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}} \\ &= K\sum_{k=\max\{1,N-K-k\}}^{\min\{n,K\}} \frac{\binom{K-1}{k-1}\binom{(N-1)-(K-1)}{(n-1)-(k-1)}}{\binom{N}{n}} \\ &= K\frac{1}{\binom{N}{n}} \sum_{u=\max\{0,N-K-k-1\}}^{\min\{n-1,K-1\}} \binom{K-1}{u}\binom{(N-1)-(K-1)}{(n-1)-u} \\ &= K\frac{\binom{N-1}{n-1}}{\binom{N}{n}} = n\frac{K}{N}.\end{aligned}$$

where we have used the sum we calculated in the first part for a $Hyp(N-1, k-1, n-1)$.

ii. We see that $X = \sum_{j=1}^{n} \mathbf{1}_{B_j}$ where $B_j$ is "in the $n$-th drawing we take a red ball". We also have

$$\mathbb{P}\left(B_j\right) = \mathbb{P}\left(B_1\right) = K/N,$$

and since $\sum_{j=1}^{n} \mathbf{1}_{B_j} = X$ we conclude

$$\mathbb{E}(X) = \mathbb{E}\left[\sum_{j=1}^{n} \mathbf{1}_{B_j}\right] = n\mathbb{P}(B_1) = n\frac{K}{N}.$$

**Q2.** **(a)** Let $X \sim \mathcal{E}(\lambda)$ be the exponential random variable with parameter $\lambda$. Compute the $\alpha$-quantile for all $\alpha \in (0,1)$.

**(b)** Let $U$ be uniformly distributed random variable in $\{1, 2, \cdots, N\}$. Determine the CDF of $U$, compute the $\alpha$-quantile for all $\alpha \in (0,1)$. For which $\alpha$ the quantile is uniquely determined?

**Solution:**

**(a)** The distribution function of an exponetially distributed random variable $X$ is given by

$$F_X(x) = \mathbb{P}(X \leq x) = 1 - e^{-\lambda x}.$$

For a $\alpha$-quantile $q_\alpha$ one has

$$\mathbb{P}(X \leq q_\alpha) \geq \alpha \text{ and } \mathbb{P}(X \geq q_\alpha) \geq 1 - \alpha.$$

Since $X$ has density (so $\mathbb{P}(X = x) = 0$ for every $x$),

$$1 - e^{-\lambda q_\alpha} = \mathbb{P}(X \leq q_\alpha) = \alpha.$$

That is

$$q_\alpha = -\ln(1-\alpha)/\lambda.$$

**(b)** The distribution function is

$$F_U(u) = \mathbb{P}(U \leq u) = \begin{cases} 0, & u < 1 \\ k/N, & u \in [k, k+1),\ k \in \{1, \cdots N-1\} \\ 1, & u \geq N \end{cases}.$$

For quantile, if $\alpha = k/N$ for some $k \in \{1, \cdots, N-1\}$, then the quantile can be chosen in $[k, k+1)$.
If $\alpha \in (k/N, (k+1)/N)$, then the quantile is uniquely determined: $q_\alpha = k+1$.

**Q3.** Let $X$ and $Y$ be random variables with joint density distribution given by

$$f_{X,Y}(x,y) = e^{-x^2 y} \mathbf{1}_{\{x \geq 1\}} \mathbf{1}_{\{y \geq 0\}}$$

**(a)** Why is this a probability measure?

**(b)** What is the density function of $X$.

**(c)** Compute $\mathbb{P}(Y \leq 1/X^2)$.

**Solution:**

**(a)** We just have to prove that $\int_{\mathbb{R}^2} f_{X,Y}(x,y)dxdy = 1$.

$$\int_1^\infty \int_0^\infty e^{-x^2 y} dy dx = \int_1^\infty \left( -\frac{e^{-x^2 y}}{x^2} \right) \Bigg|_0^\infty dx$$
$$= \int_1^\infty \frac{1}{x^2} dx = 1.$$

**(b)** We now that $f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y)dy$, then

$$f_X(x) = \int_0^\infty e^{-x^2 y} dy \mathbf{1}_{\{x \geq 1\}} = \frac{1}{x^2} \mathbf{1}_{\{x \geq 1\}}.$$

**(c)** Let's compute:

$$\mathbb{P}\left(Y \leq \frac{1}{X^2}\right) = \int_1^\infty \int_0^{1/x^2} e^{-x^2 y} dx dy$$
$$= \int_1^\infty \frac{1}{x^2} - e^{-1}\frac{1}{x^2} = 1 - e^{-1}.$$

**Q4.** Let $X$ and $Y$ be two independent exponential random variables with parameter $\lambda$. Let $a > 0$.

**(a)** What is the joint density of the couple of random variables $(X, X+Y)$?

**(b)** Let $b \leq a$, what is the probability of $X \leq b$ conditioned on the event

$$B := \{X \leq a \text{ and } X + Y \geq a\}.$$

**(c)** What is the conditional density of $X$ given the event $B$?

**Solution**

**(a)** $X \sim \mathcal{E}(\lambda)$ so it has density

$$f_X(x) = \mathbf{1}_{x\geq 0}\lambda e^{-\lambda x}$$

as well as for $Y$,

$$f_Y(y) = \mathbf{1}_{y\geq 0}\lambda e^{-\lambda y}.$$

The joint density of $(X, Z) := (X, X+Y)$ is

$$\begin{aligned} f_{X,Z}(x,z) &= f_{X,Y}(x, z-x) \\ \text{(independence of } X \text{ and } Y) &= f_X(x) f_Y(z-x) \\ &= \mathbf{1}_{x\geq 0}\lambda e^{-\lambda x}\mathbf{1}_{z-x\geq 0}\lambda e^{-\lambda(z-x)} \\ &= \mathbf{1}_{0\leq x\leq z}\lambda^2 e^{-\lambda z} \end{aligned}$$

**(b)** The event $B$ has probability:

$$\begin{aligned} \mathbb{P}(B) &= \int_a^\infty \int_0^a f_{X,Z}(x,z) dx dz \\ &= \int_a^\infty \int_0^a \mathbf{1}_{0\leq x\leq z}\lambda^2 e^{-\lambda z} dx dz \\ &= a\lambda^2 \int_a^\infty e^{-\lambda z} dz \\ &= a\lambda e^{-\lambda a}. \end{aligned}$$

Note that it is also the probability that a Poisson$(a\lambda)$ random variable equals to 1 (it is also the probability of a Poisson process with intensity $\lambda$ having exactly one point on the interval $[0, a]$). Similarly, for $0 \le b \le a$,

$$\begin{aligned}\mathbb{P}(X \le b \,|\, B) &= \frac{\mathbb{P}(X \le b, Z \ge a)}{\mathbb{P}(B)} \\ &= \frac{\int_a^\infty \int_0^b f_{X,Z}(x,z)dxdz}{a\lambda e^{-\lambda a}} \\ &= \frac{b}{a}.\end{aligned}$$

**(c)** The density is obtained by taking the derivative of $\mathbb{P}(X \le b \,|\, B)$ with respect to $b$, which gives

$$f_{X|B}(x) = \mathbf{1}_{0\le x\le a}\frac{1}{a}.$$

We conclude that the conditional law of $X$ given $B$ is the uniform distribution on $[0, a]$.

**Q5.** **(a)** Take $X$ a random variable. Prove that for all $\lambda \ge 0$

$$\mathbb{P}(X \ge t) \le \exp(-\lambda t)\mathbb{E}\left(\exp(\lambda X)\right).$$

**(b)** Define $\phi_X(\lambda) := \ln\left(\mathbb{E}\left(e^{\lambda X}\right)\right)$. Prove that $\phi(\lambda) \ge \lambda \mathbb{E}\left(X\right)$.

**(c)** Prove that

$$\mathbb{P}(X \ge t) \le \exp(-\sup_{\lambda\ge 0}\{\lambda t - \phi_X(\lambda)\}).$$

**(d)** If $X \sim N(0, \sigma)$, calculate $\phi_X(\lambda)$.

**(e)** Prove that if $X$ is a positive random variable

$$\mathbb{E}\left(X\right) = \int_0^\infty \mathbb{P}(X \ge t)dt$$

**(f)** Show that if $X \sim N(0, \sigma)$ and $Y$ is a random variable such that $\phi_Y(\lambda) \le \phi_X(\lambda)$, then

$$\mathbb{E}(Y^2) \le 4\sigma^2.$$

**Solution:**

**(a)** For $\lambda = 0$ the property is trivial. If $\lambda > 0$, using Markov (Tchebyshev) inequality we have that

$$\begin{aligned}\mathbb{P}(X \ge t) &= \mathbb{P}(\exp(\lambda X) \ge \exp(\lambda t)) \\ &\le \frac{1}{\exp(\lambda t)}\mathbb{E}\left(\exp(\lambda X)\right).\end{aligned}$$

**(b)** Given that $\exp(\lambda\cdot)$ is a convex function, thanks to Jensen inequality (Thm 3.6) we have that:

$$\begin{aligned}&\mathbb{E}(\exp(\lambda X)) \ge \exp(\mathbb{E}(\lambda X)) \\ \Rightarrow &\phi(\lambda) = \ln\left(\mathbb{E}(\exp(\lambda X))\right) \ge \lambda\mathbb{E}\left(X\right).\end{aligned}$$

**(c)** Using part a) we have that for all $\lambda > 0$

$$\begin{aligned} &\mathbb{P}(X \geq \lambda) \leq \exp(-\lambda t)\mathbb{E}\left(e^{\lambda X}\right) = \exp(\phi_X(\lambda) - \lambda t) \\ \Rightarrow &\mathbb{P}(X \geq \lambda) \leq \exp(-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}), \end{aligned}$$

where we have just taken the infimum of the exponential term over $\lambda \geq 0$.

**(d)** Let's compute

$$\begin{aligned} \mathbb{E}\left(e^{\lambda X}\right) &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp(\lambda x)\exp(\frac{-x^2}{2\sigma^2})dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp(-\frac{x^2 - 2\lambda\sigma^2}{2\sigma^2})dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp(\lambda x)\exp(-\frac{(x-\lambda\sigma^2)^2}{2\sigma^2})\exp(\frac{\lambda^2\sigma^2}{2})dx \\ &= \exp(\frac{\lambda^2\sigma^2}{2}), \end{aligned}$$

then $\phi(\lambda) = \frac{1}{2}\lambda^2\sigma^2$.

**(e)** We just have to use Fubini's Theorem to compute

$$\begin{aligned} \mathbb{E}\left(X\right) &= \mathbb{E}\left(\int_0^{\infty} \mathbf{1}_{\{t \leq X\}} dt\right) \\ &= \int_0^{\infty} \mathbb{E}\left(\mathbf{1}_{\{X \geq t\}}\right) dt \\ &= \int_0^{\infty} \mathbb{P}(X \geq t)dt. \end{aligned}$$

**(f)** First let's compute:

$$\sup_{\lambda > 0}\left\{\lambda t - \frac{1}{2}\lambda^2\sigma^2\right\},$$

this is just a quadratic function on $\lambda$, so it attains it maximum in $\lambda_{\max} = t/\sigma^2 > 0$. Then its supremum is just $t^2/2\sigma^2$. So thanks to problem c) we have that

$$\begin{aligned} \mathbb{P}(Y \geq t) &\leq \exp(-\sup_{\lambda \geq 0}\{\lambda t - \phi_Y(\lambda)\}) \\ &\leq \exp(-\sup_{\lambda \geq 0}\{\lambda t - \phi_X(\lambda)\}) \\ &= \exp\left(-\frac{t^2}{2\sigma^2}\right). \end{aligned}$$

Thus, thanks to part e) we have that

$$\begin{aligned}\mathbb{E}\left(Y^2\right) &= \int_0^\infty P(Y^2 \geq t)dt = \int_0^\infty P(Y \geq \sqrt{t})dt + \int_0^\infty P(-Y \geq \sqrt{t})dt \\ &\leq 2\int_0^\infty \exp\left(-\frac{t}{2\sigma^2}\right)dt \\ &= 4\sigma^2.\end{aligned}$$

Where we have used also

$$\phi_{-Y}(\lambda) = \phi_Y(-\lambda) \leq \phi_X(-\lambda) = \phi_{-X}(\lambda) = \phi_X(\lambda),$$

hence the previous step can also be applied to $\mathbb{P}(-Y \geq \sqrt{t})$.