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Solution Series 7

Q1. Suppose two random variables X_1 and X_2 have a continuous joint distribution for which the joint p.d.f. is as follows:

$$f(x_1, x_2) = \begin{cases} 4x_1x_2 & \text{for } 0 < x_1, x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $Cov(X_1, X_2)$. Determine the joint p.d.f. of two new random variables Y_1 and Y_2, which are defined by the relations

$$Y_1 = \frac{X_1}{X_2} \text{ and } Y_2 = X_1X_2.$$

Solution:

We apply the formula

$$Cov(X_1, X_2) = \mathbb{E}(X_1X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2).$$

Since the marginal density for X_1 is

$$f_1(x_1) = \int_0^1 4x_1x_2dx_2 = 2x_1,$$

$$\mathbb{E}(X_1) = \int_0^1 x_1f_1(x_1)dx_1 = 2/3 = \mathbb{E}(X_2).$$

We have also

$$\mathbb{E}(X_1X_2) = \int_0^1 \int_0^1 x_1x_2 4x_1x_2dx_1dx_2 = 4/9.$$

Thus $Cov(X_1, X_2) = 0$. Actually, one can see from $f_1(x_1)f_2(x_2) = f(x_1, x_2)$ that X_1 and X_2 are independent.

Joint p.d.f. of $(Y_1, Y_2) = (r_1(X_1, X_2), r_2(X_1, X_2))$: where $r_1(x_1, x_2) = x_1/x_2$ and $r_2(x_1, x_2) = x_1x_2$. We have that

$$x_1 = \sqrt{r_1r_2} \text{ and } x_2 = \sqrt{r_2/r_1}.$$

$$Jac(r_1, r_2) = det\begin{pmatrix} \partial x_1/\partial r_1 & \partial x_2/\partial r_1 \\ \partial x_1/\partial r_2 & \partial x_2/\partial r_2 \end{pmatrix} = \frac{1}{4}det\begin{pmatrix} \sqrt{r_2/r_1} & -\sqrt{r_2/r_1^3} \\ \sqrt{r_1/r_2} & \sqrt{1/r_1r_2} \end{pmatrix} = \frac{1}{2r_1}.$$

By the transformation formula of the density,

$$\begin{aligned} f_{Y_1,Y_2}(r_1,r_2) &= f_{X_1,X_2}(\sqrt{r_1 r_2}, \sqrt{r_2/r_1}) Jac(r_1,r_2) \\ &= 4r_2/2r_1 \mathbf{1}_{0<\sqrt{r_1r_2},\sqrt{r_2/r_1}<1} \\ &= \mathbf{1}_{0<\sqrt{r_1r_2},\sqrt{r_2/r_1}<1} 2r_2/r_1. \end{aligned}$$

Let α and β be positive numbers. A random variable X has the *gamma distribution with parameters α and β* if X has a continuous distribution for which the p.d.f. is

$$f(x|\alpha,\beta) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

Where Γ is the function defined as: for $\alpha > 0$,

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

Q2. Let X have the gamma distribution with parameters α and β.

(a) For $k = 1, 2, \ldots$, show that the k-th moment of X is

$$\mathbb{E}(X^k) = \frac{\Gamma(\alpha+k)}{\beta^k \Gamma(\alpha)} = \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{\beta^k}.$$

What are $\mathbb{E}(X)$ and $Var(X)$?

(b) What is the moment generating function of X?

(c) If the random variables $X_1, \cdots, X_k$ are independent, and if X_i (for $i = 1, \ldots, k$) has the gamma distribution with parameters α_i and β, show that the sum $X_1 + \cdots + X_k$ has the gamma distribution with parameters $\alpha_1 + \cdots + \alpha_k$ and β.

Solution:

(a) For $k = 1, 2, \cdots$

$$\begin{aligned} \mathbb{E}[X^k] &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^k x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+k-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \beta^{-\alpha-k} y^{\alpha+k-1} e^{-y} dy \\ &= \frac{\Gamma(\alpha+k)}{\beta^k \Gamma(\alpha)}, \end{aligned}$$

where the change of variable $y = \beta x$ is used. It can be easily seen that for $\alpha > 0$,

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

We then deduce the second equality.

$$\mathbb{E}(X) = \alpha/\beta$$
$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \alpha/\beta^2.$$

(b) By definition of moment generating function,

$$\begin{aligned}\psi_X(t) &= \mathbb{E}(e^{tX}) \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} x^{\alpha-1} e^{-\beta x} dx \\ &= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} dx \\ &= \left(\frac{\beta}{\beta - t}\right)^\alpha\end{aligned}$$

The above computation is only valid for $t < \beta$.

(c) If ψ_i denotes the m.g.f of X_i, then it follows from the last question that for $i = 1, \cdots, k$,

$$\psi_i(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha_i}.$$

The m.g.f. ψ of $X_1 + \cdots + X_k$ is, by independence,

$$\psi(t) = \prod_{i=1}^k \psi_i(t) = \left(\frac{\beta}{\beta - t}\right)^{\alpha_1 + \cdots + \alpha_k} \qquad \text{for } t < \beta.$$

It coincides with the m.g.f. of a Gamma random variable with parameter $(\alpha_1 + \cdots + \alpha_k, \beta)$, in an open interval of 0, thus the sum $X_1 + \cdots + X_k$ has the Gamma distribution.

Q3. Service Times in a Queue. For $i = 1, \cdots, n$, suppose that customer i in a queue must wait time X_i for service once reaching the head of the queue. Let Z be the rate at which the average customer is served. A typical probability model for this situation is to say that, conditional on $Z = z$, $X_1, \ldots, X_n$ are i.i.d. with a distribution having the conditional p.d.f. $g_1(x_i|z) = z \exp(-zx_i)$ for $x_i > 0$. Suppose that Z is also unknown and has the p.d.f. $f_2(z) = 2\exp(-2z)$ for $z > 0$.

(a) What is the joint p.d.f. of $X_1, \ldots, X_n, Z$.

(b) What is the marginal joint distribution of $X_1, \ldots, X_n$.

(c) What is the conditional p.d.f. $g_2(z|x_1, \ldots, x_n)$ of Z given $X_1 = x_i, \ldots, X_n = x_n$? For this we can set $y = 2 + \sum_{i=1}^n x_i$.

(d) What is the expected average service rate given the observations $X_1 = x_1, \cdots, X_n = x_n$?

Solution:

(a) The joint p.d.f. of $X_1, \cdots, X_n, Z$ is

$$\begin{aligned} f(x_1, \cdots, x_n, z) &= \prod_{i=1}^{n} g_1(x_i|z) f_2(z) \\ &= 2z^n \exp(-z[2 + x_1 + \cdots + x_n]), \end{aligned}$$

if $z, x_1, \cdots, x_n > 0$ and 0 otherwise.

(b) The marginal joint distribution of $X_1, \cdots, X_n$ is obtained by integrating z out of the joint p.d.f. above.

$$\int_0^\infty f(x_1, \cdots, x_n, z) dz = \frac{2\Gamma(n+1)}{(2 + x_1 + \cdots + x_n)^{n+1}} = \frac{2(n!)}{(2 + x_1 + \cdots + x_n)^{n+1}},$$

for all $x_i > 0$ and 0 otherwise.

(c) We set $y = 2 + \sum_{i=1}^n x_i$, for $z > 0$

$$\begin{aligned} g_2(z|x_1, \cdots, x_n) &= f(x_1, \cdots, x_n, z) \frac{y^{n+1}}{2(n!)} \\ &= \frac{z^n \exp(-zy) y^{n+1}}{n!} \\ &= \frac{y^{n+1}}{\Gamma(n+1)} z^{n+1-1} e^{-yz}, \end{aligned}$$

we recognize the conditional distribution of Z given $X_1 = x_1, \cdots X_n = x_n$ is Gamma distribution with parameter $\alpha = n + 1$, $\beta = y$.

(d) The conditional expected value of Z given $X_1 = x_1, \cdots X_n = x_n$ is the expected value of Gamma distribution with parameter $\alpha = n + 1$, $\beta = y$, which by Q2.a, equals to

$$\mathbb{E}(Z|X_1 = x_1, \cdots X_n = x_n) = \frac{n+1}{2 + \sum_{i=1}^n x_i}.$$

Q4. LEAST-SQUARES LINE.

(a) Let $(x_1, y_1), \ldots, (x_n, y_n)$ be a set of n points of $\mathbb{R}^2$ and x_is are not all the same. Show that the straight line defined by the equation $y(x) = \hat{\beta}_0 + \hat{\beta}_1 x$ that minimizes the sum of the squares of the vertical deviations of all the points from the line has the following slope and intercept, i.e. $(\hat{\beta}_0, \hat{\beta}_1)$ minimizes

$$I(\beta_0, \beta_1) := \sum_{i=1}^{n} (\beta_0 + \beta_1 x_i - y_i)^2$$

over all choices of $(\beta_0, \beta_1) \in \mathbb{R}^2$:

$$\begin{aligned} \hat{\beta}_1 &= \frac{\sum_{i=1}^n (y_i - \bar{y})(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}, \\ \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x}, \end{aligned}$$

Table 1: Data for Q4.(b)

i	x_i	y_i
1	0.5	40
2	1.0	41
3	1.5	43
4	2.0	42
5	2.5	44
6	3.0	42
7	3.5	43
8	4.0	42

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.
The minimizing line is called the *least-squares line*. Remark that the least-squares line passes through the point $(\bar{x}, \bar{y})$.

(b) Fit a straight line of the form $y = \beta_0 + \beta_1 x$ to these values by the method of least squares (with your calculator or Excel).

Solution:

(a) Using the fact that $I(\beta_0, \beta_1) \to +\infty$ as $\|(\beta_0, \beta_1)\| \to \infty$ (which is true since the x_is are not all the same), the infimum of I is in approximated in some compact set. Since I is continuous, the infimum of I is a minimum. We can look for critical points $(\hat{\beta}_0, \hat{\beta}_1)$:

$$\partial_{\beta_0} I(\hat{\beta}_0, \hat{\beta}_1) = 2\sum_{i=1}^{n} \hat{\beta}_0 + \hat{\beta}_1 x_i - y_i = 0$$
$$\partial_{\beta_1} I(\hat{\beta}_0, \hat{\beta}_1) = 2\sum_{i=1}^{n} x_i(\hat{\beta}_0 + \hat{\beta}_1 x_i - y_i) = 0.$$

We solve the above system:

$$n\hat{\beta}_0 + n\bar{x}\hat{\beta}_1 = n\bar{y} \quad \Rightarrow \quad \hat{\beta}_0 = \bar{y} - \bar{x}\hat{\beta}_1,$$

and

$$
\begin{aligned}
& n\bar{x}\hat{\beta}_0 + \left(\sum_{i=1}^{n} x_i^2\right)\hat{\beta}_1 = \sum_{i=1}^{n} x_i y_i \\
\Rightarrow \quad & n\bar{x}(\bar{y} - \bar{x}\hat{\beta}_1) + \left(\sum_{i=1}^{n} x_i^2\right)\hat{\beta}_1 = \sum_{i=1}^{n} x_i y_i \\
\Rightarrow \quad & \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right)\hat{\beta}_1 = \sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} \\
\Rightarrow \quad & \left(\sum_{i=1}^{n} (x_i - \bar{x})^2\right)\hat{\beta}_1 = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \\
\Rightarrow \quad & \hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}.
\end{aligned}
$$

(b) We apply the above formula to find $\hat{\beta}_1 = 40.89$ and $\hat{\beta}_0 = 0.55$.