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Solution Series 8

Q1. If $\log(X)$ has the normal distribution with mean μ and variance σ^2, we say that X has the *lognormal distribution* with parameters μ and σ^2.

A popular model for the change in the price of a stock over a period of time of length u is to say that the price after time u is $S_u = S_0 Z_u$, where Z_u has the lognormal distribution with parameter μu and $\sigma^2 u$. In this formula, S_0 is the present price of the stock, and σ is called the *volatility* of the stock price.

(a) What is the expected value of S_1?

(b) Find the distribution of $1/S_1$.

(c) What is the expected value of $1/S_1$?

(d) What are k-th moments of S_1, for $k = 1, 2, \dots$?

Solution:

(a) By definition of a lognormal distribution, $N := \log(Z_1) \sim \mathcal{N}(\mu, \sigma^2)$. Thus

$$\mathbb{E}(Z_1) = \mathbb{E}(e^N) = \varphi_N(1) = e^{\mu+\sigma^2/2},$$

where we have used the moment generating function of N:

$$\varphi_N(t) = \mathbb{E}(e^{tN}) = e^{t\mu+t^2\sigma^2/2}.$$

We get:

$$\mathbb{E}(S_1) = S_0 e^{\mu+\sigma^2/2}.$$

(b) Let $X := (\log(Z_1) - \mu)/\sigma \sim \mathcal{N}(0, 1)$,

$$S_1 = S_0 e^{\mu+\sigma X} \Rightarrow \frac{1}{S_1} = \frac{e^{-\mu-\sigma X}}{S_0}.$$

Hence $1/S_1$ has the distribution of $1/S_0$ times a lognormal random variable with parameter $-\mu$ and σ^2.

(c) By question a)

$$\mathbb{E}(1/S_1) = e^{-\mu+\sigma^2/2}/S_0.$$

Note that if we use a random variable to modelize the rate of change from A to B, than its inverse will be the rate of change from B to A. The product of expected value is always larger than 1 if the variance is non-zero (Jensen's inequality). Here $\mathbb{E}(S_1)\mathbb{E}(1/S_1) = e^{\sigma^2}$.

(d) The k-th moment of a lognormal distribution is easy to compute: For $k = 1, 2, \cdots$,

$$\mathbb{E}(S_1^k) = S_0^k \mathbb{E}(e^{k\mu + k\sigma X}) = (S_0 e^{\mu})^k e^{k^2\sigma^2/2}.$$

Q2. Suppose that Z has the standard normal distribution, V has the χ-squared distribution with n degrees of freedom, and that Z and V are independent. Let

$$T = \frac{Z}{\sqrt{V/n}}.$$

You will show that T has p.d.f. given by

$$f(t) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\Gamma(n/2)} \left(1 + \frac{t^2}{n}\right)^{-(n+1)/2} \qquad t \in \mathbb{R}.$$

Recall that we have seen in Series 4, that the p.d.f. of a χ-squared with n degrees of freedom is, for some $c_n \in \mathbb{R}$:

$$f_V(x) = c_n x^{n/2-1} e^{-x/2} \mathbf{1}_{\{x \geq 0\}}.$$

(a) Find the joint p.d.f. of (T, V).

(b) Show first that the conditional distribution of T given $V = v$ is normal with mean 0 and variance $\frac{n}{v}$.

(c) Compute c_n.

(d) Find the p.d.f. of T.

Solution:

(a) Since Z and V are independent, the p.d.f. of the couple (Z, V) is

$$f_{Z,V}(z, v) = f_Z(z) f_V(v),$$

where $f_Z(z) = e^{-z^2/2}/\sqrt{2\pi}$ is the p.d.f. of standard normal distribution. By the transformation formula, one obtains the joint p.d.f. of (T, V):

$$f_{T,V}(t, v) = f_Z\left(t\sqrt{\frac{v}{n}}\right) f_V(v) \sqrt{\frac{v}{n}}.$$

(b) The conditional probability given $V = v$ is

$$f_{T|v}(t) = \frac{f_{T,V}(t, v)}{f_V(v)} = f_Z\left(t\sqrt{\frac{v}{n}}\right)\sqrt{\frac{v}{n}} = \exp\left(-\frac{t^2}{2n/v}\right)\frac{\sqrt{v/n}}{\sqrt{2\pi}},$$

which is the p.d.f. of a centered normal distribution with variance n/v. One can guess it by replace directly V by v in the expression of T (which can be proven when Z and V are independent).

(c) The c_n can be computed as the constant making f_V a p.d.f. (with integral 1):

$$1/c_n = \int_0^\infty x^{n/2-1}\exp(-x/2)dx = 2^{n/2}\Gamma(n/2).$$

(d) The p.d.f. of T is obtained by integrating $f_{T,V}$:

$$\begin{aligned} f_T(t) &= \int_0^\infty f_{T,V}(t,v)dv \\ &= \frac{c_n}{\sqrt{2\pi n}}\int_0^\infty \exp\left(-v\frac{t^2}{2n}\right) v^{n/2-1}e^{-v/2}\sqrt{v}dv \\ &= \frac{c_n}{\sqrt{2\pi n}}\int_0^\infty \exp\left(-v\left(\frac{t^2}{2n}+\frac{1}{2}\right)\right) v^{(n+1)/2-1}dv \\ &= \frac{c_n}{\sqrt{2\pi n}}\frac{\Gamma((n+1)/2)}{\left(\frac{t^2}{2n}+\frac{1}{2}\right)^{(n+1)/2}} \\ &= \left(\frac{t^2}{n}+1\right)^{-(n+1)/2}\frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{\pi n}}2^{-n/2-1/2+(n+1)/2} \\ &= \left(\frac{t^2}{n}+1\right)^{-(n+1)/2}\frac{\Gamma((n+1)/2)}{\Gamma(n/2)\sqrt{\pi n}}. \end{aligned}$$

Q3. We would like to compute

$$A := \int_{-3}^1 \frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

using Monte-Carlo method:

(a) Express A under the form $\mathbb{E}(f(U))$, where U is a standard Gaussian random variable, and f an appropriate function.

(b) Take $(U_i)_{i\in\mathbb{N}}$ an i.i.d. family having the same law as U. Set

$$S_n := \frac{1}{n}\sum_{i=1}^n f(U_i).$$

What is the distribution of $S_n - A$?

(c) Compute $\mathbb{E}(S_n)$ and show that $Var(S_n) = (A - A^2)/n$.

(d) Show that for any $x > 0$, $\mathbb{P}(|S_n - A| \geq x) \leq 1/nx^2$, thus converges to 0 when $n \to \infty$.

(e) Which theorem can you apply to get directly the above convergence?

Solution:

(a) Set $f = \mathbf{1}_{[-3,1]}$, then

$$A = \mathbb{E}[f(U)] = \mathbb{P}(U \in [-3,1]) = \mathbb{P}(f(U) = 1).$$

(b) nS_n follows the binomial law with parameter (n, A). Hence

$$\mathbb{P}(nS_n = k) = \binom{n}{k} A^k (1-A)^{n-k},$$

or equivalently

$$\mathbb{P}(S_n - A = k/n - A) = \binom{n}{k} A^k (1-A)^{n-k}.$$

(c) $\mathbb{E}(S_n) = A$ and

$$\begin{aligned} Var(S_n) &= nVar(f(U_i)/n) \\ &= (A - A^2)/n. \end{aligned}$$

(d) By Tchebychev inequality

$$\mathbb{P}(|S_n - A| \geq x) = \mathbb{P}(|S_n - A|^2 \geq x^2) \leq \frac{Var(S_n)}{x^2} \leq \frac{1}{nx^2}.$$

The convergence is valid for all $x > 0$, which means that S_n converges in probability to A. To numerically approximate the value A, one can sample independently a family $(U_i)_{i=1..n}$ having the standard normal distribution, and counts the number of points in the interval $[-3, 1]$ then divide by n. While the n becomes larger we get a better approximation of A.

(e) We can apply the weak Law of large number.

Q4. *Fitting a polynomial by Methode of Least Squares* Suppose now that instead of simply fitting a straight line to n plotted points, we wish to fit a polynomial of degree k ($k \geq 2$). such a polynomial will have the following form:

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots + \beta_k x^k.$$

The method of least squares specifies that the constants $\beta_0, \cdots, \beta_k$ should be chosen that the sum

$$Q(\beta_0, \cdots, \beta_k) = \sum_{i=1}^{n} [y_i - (\beta_0 + \beta_1 x_i + \cdots + \beta_k x_i^k)]^2$$

of the squares of the vertical deviations of the points from the curve is a minimum.

(a) Which equation system should a minimizer $\hat{\beta}_0, \cdots, \hat{\beta}_k$ satisfy?

(b) Fit a parabola (polynomial of degree 2) to the 10 points given in the table.

Solution:

Table 1: Data for Q4.(b)

i	x_i	y_i
1	1.9	0.7
2	0.8	-1.0
3	1.1	-0.2
4	0.1	-1.2
5	-0.1	-0.1
6	4.4	3.4
7	4.6	0.0
8	1.6	0.8
9	5.5	3.7
10	3.4	2.0

(a) If we calculate the $k+1$ partial derivatives $\partial Q/\partial\beta_0, \cdots, \partial Q/\partial\beta_k$, and we set each of these derivatives equal to 0, we obtain the following $k+1$ linear equations involving $k+1$ unknown values $\beta_0, \cdots, \beta_k$:

$$\begin{aligned}
\hat{\beta}_0 n + \hat{\beta}_1 \sum_{i=1}^n x_i + \cdots + \hat{\beta}_k \sum_{i=1}^n x_i^k &= \sum_{i=1}^n y_i, \\
\hat{\beta}_0 \sum_{i=1}^n x_i + \hat{\beta}_1 \sum_{i=1}^n x_i^2 + \cdots + \hat{\beta}_k \sum_{i=1}^n x_i^{k+1} &= \sum_{i=1}^n x_i y_i, \\
&\vdots \\
\hat{\beta}_0 \sum_{i=1}^n x_i^k + \hat{\beta}_1 \sum_{i=1}^n x_i^{k+1} + \cdots + \hat{\beta}_k \sum_{i=1}^n x_i^{2k} &= \sum_{i=1}^n x_i^k y_i.
\end{aligned}$$

As before, if these equations have a unique solution, that solution provides the minimum value for Q. A necessary and sufficient condition for a unique solution is that the determinant of the $(k+1)\times(k+1)$ matrix formed by the coefficients of $\hat{\beta}_0, \cdots, \hat{\beta}_k$ above is not zero.

(b) In this example, it is found that the equations are

$$\begin{aligned}
10\beta_0 + 23.3\beta_1 + 90.37\beta_2 &= 8.1, \\
23.3\beta_0 + 90.37\beta_1 + 401.0\beta_2 &= 43.59, \\
90.37\beta_0 + 401.0\beta_1 + 1892.7\beta_2 &= 204.55.
\end{aligned}$$

The unique solution is

$$\hat{\beta}_0 = -0.744, \quad \hat{\beta}_1 = 0.616, \quad \hat{\beta}_2 = 0.013.$$

Hence the least squares parabola is

$$y = -0.744 + 0.616x + 0.013x^2.$$