7.1. Mehrfache Integrale

(a)

$$\begin{aligned}
\int_0^2 \int_y^{2y} (x + e^y)\,dx\,dy &= \int_0^2 \left(\int_y^{2y} (x+e^y)\,dx \right) dx \\
&= \int_0^2 \left(\frac{1}{2}x^2 + e^y x \Big|_y^{2y} \right) dy = \int_0^2 \left(\frac{3}{2}y^2 + ye^y \right) dy \\
&= \frac{3}{2}\int_0^2 y^2\,dy + \int_0^2 ye^y\,dy \;=\; \frac{3}{2}\cdot \frac{y^3}{3}\Big|_0^2 + \int_0^2 y(e^y)'\,dy \\
&= 4 + \left(ye^y\Big|_0^2 - \int_0^2 e^y\,dy \right) = 4 + 2e^2 - (e^2 - e^0) = 5 + e^2\,.
\end{aligned}$$

(b)

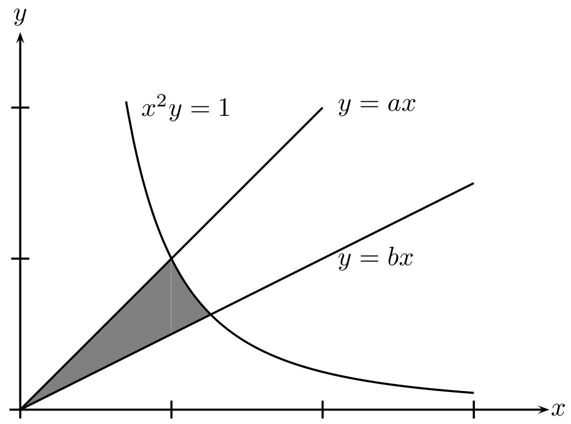
$$\begin{aligned}
\int_1^2 \int_{\sqrt[3]{y}}^{y} x^2 y^3\,dx\,dy &= \int_1^2 \left(\frac{x^3}{3} y^3 \right)\Big|_{x=y^{1/3}}^{y} dy = \frac{1}{3}\int_1^2 \left(y^3\cdot y^3 - y \cdot y^3 \right) dy \\
&= \frac{1}{3}\int_1^2 \left(y^6 - y^4 \right) dy = \frac{1}{3}\left(\frac{y^7}{7} - \frac{y^5}{5} \right)\Big|_1^2 \\
&= \frac{1}{3}\left(\frac{1}{7}(2^7-1) - \frac{1}{5}(2^5-1) \right) = \frac{1}{3}\left(\frac{127}{7} - \frac{31}{5} \right) = \frac{418}{105}\,.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{-\pi}^{\pi} \int_0^{\pi^2 - y^2} \cos y\,dx\,dy &= \int_{-\pi}^{\pi} \pi^2 \cos y\,dy - \int_{-\pi}^{\pi} y^2 \cos y\,dy \\
&= \pi^2 \sin y\Big|_{-\pi}^{\pi} - \left(y^2 \sin y\Big|_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi} y \sin y\,dy \right) \\
&= -2y\cos y\Big|_{-\pi}^{\pi} + 2\int_{-\pi}^{\pi} \cos y\,dy = 2(\pi + \pi) = 4\pi\,.
\end{aligned}$$

7.2. Mehrfache Integrale

Wir betrachten das Gebiet B



Wir berechnen das Doppelintegral

$$\int_B xy\,dxy\,.$$

Dazu berechnen wir die Schnittpunkte $P_1\,, P_2$

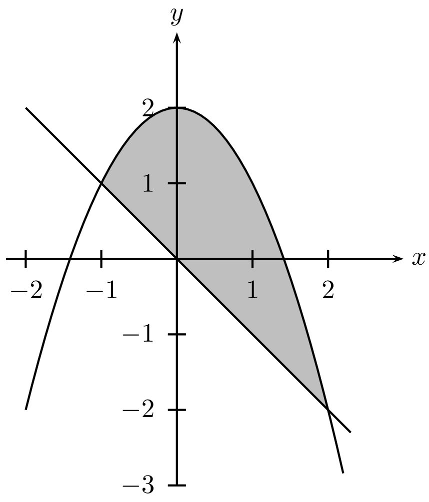
$$P_1 = \left(a^{-\frac{1}{3}}, a^{\frac{2}{3}}\right) \quad \text{und} \quad P_2\left(b^{-\frac{1}{3}}, b^{\frac{2}{3}}\right)\,.$$

Damit folgt

$$\begin{aligned}
\int_B xy\,dxy &= \int_0^{a^{-\frac{1}{3}}} \left(\int_{bx}^{ax} xy\,dy\right) dx + \int_{a^{-\frac{1}{3}}}^{b^{-\frac{1}{3}}} \left(\int_{bx}^{\frac{1}{x^2}} xy\,dy\right) dx \\
&= \int_0^{a^{-\frac{1}{3}}} \frac{1}{2}(a^2 - b^2)x^3\,dx + \int_{a^{-\frac{1}{3}}}^{b^{-\frac{1}{3}}} \frac{x}{2}\left(\frac{1}{x^4} - (bx)^2\right) dx \\
&= \frac{1}{8}\left(a^{\frac{2}{3}} - \frac{b^2}{a^{\frac{4}{3}}}\right) + \frac{2a^2 - 3a^{\frac{4}{3}}b^{\frac{2}{3}} + b^2}{8a^{\frac{4}{3}}} \\
&= \frac{3}{8}\left(a^{\frac{2}{3}} - b^{\frac{2}{3}}\right)\,.
\end{aligned}$$

7.3. Mehrfache Integrale

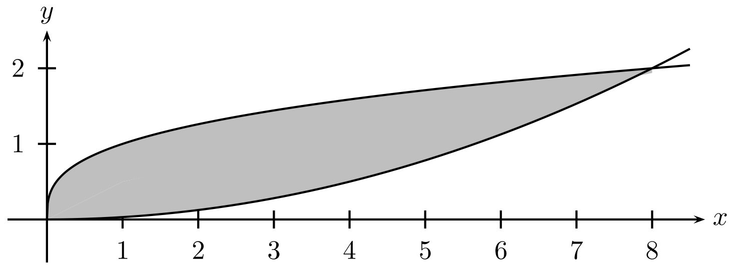
(a) Das Gebiet worüber integriert wird, sieht folgendermassen aus:



Daraus folgt:

$$\int_{-1}^{2}\left(\int_{-x}^{2-x^2} f(x,y)\,dy\right)\,dx$$
$$= \int_{-2}^{1}\left(\int_{-y}^{\sqrt{2-y}} f(x,y)\,dx\right)\,dy + \int_{1}^{2}\left(\int_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)\,dx\right)\,dy$$

(b) Das Gebiet sieht hier folgendermassen aus:



Und darum ergibt die Umkehrung der Integrationsreihenfolge:

$$\int_{0}^{2}\left(\int_{y^3}^{4\sqrt{2y}} f(x,y)\,dx\right)\,dy = \int_{0}^{8}\left(\int_{\frac{x^2}{32}}^{x^{\frac{1}{3}}} f(x,y)\,dy\right)\,dx$$

7.4. Mehrfache Integrale

Die Menge E ist z-einfach, weil

$$E = \left\{(x,y,z)\in\mathbb{R}^3 : (x,y)\in T, 0 < z < 2-x-y\right\}$$

gilt, wobei

$$\begin{aligned} T &= \Big\{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 2\Big\} \\ &= \Big\{(x,y) \in \mathbb{R}^2 : 0 < x < 2, 0 < y < 2 - x\Big\} \end{aligned}$$

ist.

Wir bekommen daher

$$\int_E (x + y - z)\, dx\, dy\, dz = \int_T \left(\int_0^{2-x-y} (x + y - z)\, dz\right) dx\, dy\,.$$

Das innere Integral ist gleich

$$\int_0^{2-x-y} (x + y - z)\, dz = \left(xz + yz - \frac{z^2}{2}\right)_{z=0}^{z=2-x-y} = 4(x + y) - \frac{3}{2}(x + y)^2 - 2\,.$$

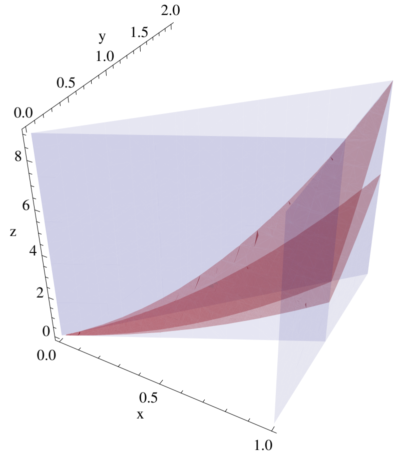
Somit erhalten wir

$$\begin{aligned} \int_E (x + y - z)\, dx\, dy\, dz &= \int_T \left(4(x + y) - \frac{3}{2}(x + y)^2 - 2\right) dx\, dy \\ &= \int_0^2 \left(\int_0^{2-x} \left[4(x + y) - \frac{3}{2}(x + y)^2 - 2\right] dy\right) dx \\ &= \int_0^2 \left[4xy + 2y^2 - \frac{3}{2}\left(x^2 y + xy^2 + \frac{1}{3}y^3\right) - 2y\right]_{y=0}^{y=2-x} dx \\ &= \int_0^2 \left(\frac{1}{2}x^3 - 2x^2 + 2x\right) dx = \left(\frac{1}{8}x^4 - \frac{2}{3}x^3 + x^2\right)_0^2 \\ &= 2 - \frac{16}{3} + 4 = \frac{2}{3}\,. \end{aligned}$$

7.5. Mehrfache Integrale

Integrationsgrenzen:

- z läuft bei festgehaltenem x und y zwischen $x^2 + y^2$ und $x^2 + 2y^2$.
- Grenzen für y und x aus dem Bild sieht man $x \in [0, 1]$ und $y \in [x, 2x]$.



$$\begin{aligned}
\text{Volumen} &= \int_0^1 \int_x^{2x} \int_{x^2+y^2}^{x^2+2y^2} 1 \, dz \, dy \, dx \\
&= \int_0^1 \int_x^{2x} \Big((x^2+2y^2) - (x^2+y^2) \Big) \, dy \, dx = \int_0^1 \int_x^{2x} y^2 \, dy \, dx \\
&= \int_0^1 \frac{y^3}{3} \Big|_{y=x}^{2x} \, dx \;=\; \int_0^1 \frac{1}{3} \Big((2x)^3 - x^3 \Big) \, dx \\
&= \int_0^1 \frac{7}{3} x^3 \, dx \;=\; \frac{7}{12} x^4 \Big|_{x=0}^{1} = \frac{7}{12} \, .
\end{aligned}$$